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ON RIGHT P.P. RINGS

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In this paper we study non-commutative p.p. rings and Baer rings in relation to their quotient rings. First, we investigate right p.p. rings and their classical quotient rings. It has been shown by several authors that a commutative ring R is a p.p. ring if and only if the classical quotient ring Q of R is von Neumann regular and every idempotent of Q lies in R ([1], [3], [19]). In Theorem 1, we extend this result to normal p.p. rings. Next, we consider a right non-singular ring R and a necessary and sufficient condition for R to be a right Utumi, Baer ring is given (Theorem 3). Using this, we define the Baer hull of a reduced right Utumi ring. In Theorem 5 (resp. Theorem 6), we characterize a ring which is a finite direct sum of prime right and left Goldie right p.p. rings (resp. finite direct sum of right Ore domains). In Theorem 7 (resp. Theorem 8) we prove that a semiprime right and left Goldie ring (resp. normal ring) R is a right p.p. ring if and only if every divisible right R-module is p-injective. Finally, we consider rings all of whose non-zero factor rings are right p.p. rings.

Throughout this paper "a ring" means "a non-zero associative ring with 1" and all modules are unital. For any ring S, we denote by $E(S)$ the set of all idempotents in S. Let R be an arbitrary ring. For a non-empty subset X of a right (resp. left) R-module M, we set $r_R(X)=\{r \in R \mid Xr=0\}$ (resp. $l_R(X)=\{r \in R \mid rX=0\}$). For a right (or left) R-module M, $Z(M)$ denotes as usual the singular submodule of M. A right R-module M is called a *c.p. module* if every cyclic submodule of M is projective, or equivalently, for any $m \in M$, there exists $e \in E(R)$ such that $r_R(m)=eR$. We call R a *right* (resp. *left*) *p.p. ring* if R_R (resp. ${}_RR$) is a c.p. module. A ring which is right and left p.p. is said to be a *p.p. ring*. R is called a *normal ring* if every idempotent of R is central. As was shown by Endo [2, Proposition 2], a normal ring R is a right p.p. ring if and only if it is a left p.p. ring and if this condition is satisfied, then R is reduced. A *right order* R in a ring Q is a subring of Q such that every element of Q has the form ab^{-1} for some $a, b \in R$. Similarly, we define a *left order* in Q.

We begin by showing that a ring Q which has a right p.p. right order is also a right p.p. ring. First, we prove the following

Lemma 1. *Let R be a right order in a ring Q. Then for any*

$ab^{-1} \in Q$ $(a, b \in R)$ we have $r_Q(ab^{-1}) = br_R(a)Q$.

Proof. It is clear that $r_Q(ab^{-1}) \supseteq br_R(a)Q$. On the other hand, let cd^{-1} $(c, d \in R)$ be an arbitrary element of $r_Q(ab^{-1})$. Since R is a right order in Q, we can write $b^{-1}c = rs^{-1}$ for some $r, s \in R$. Then r is in $r_R(a)$, and so $cd^{-1} = brs^{-1}d^{-1} \in br_R(a)Q$. This shows that $r_Q(ab^{-1}) \subseteq br_R(a)Q$.

Proposition 1. *Let R be a right order in a ring Q. If R is a right p.p. ring, then Q is also a right p.p. ring.*

Proof. Let ab^{-1} $(a, b \in R)$ be an arbitrary element of Q. Since R is a right p.p. ring, $r_R(a) = eR$ for some $e \in E(R)$. Then, by Lemma 1, $r_Q(ab^{-1}) = beQ = beb^{-1}Q$. Clearly, beb^{-1} is in $E(Q)$. Therefore Q is a right p.p. ring.

Corollary 1. *Let R be a right order in a ring Q. Suppose that R is a normal ring. Then R is a right p.p. ring if and only if Q is a right p.p. ring and $E(Q) = E(R)$.*

Proof. Assume first that R is a right p.p. ring. Then, by Proposition 1, Q is a right p.p. ring. Let e be an arbitrary element of $F(Q)$ and write $1-e = ab^{-1}$ with $a, b \in R$. Clearly, $r_Q(1-e) = eQ$. On the other hand, by Lemma 1, $r_Q(ab^{-1}) = br_R(a)Q = bfQ$, where f is an element of $E(R)$ such that $r_R(a) = fR$. Since R is normal, $r_Q(ab^{-1}) = fbQ = fQ$. Thus we have $eQ = fQ$. Now it is easy to see that $e = f \in E(R)$.

To see the converse, we assume now that Q is a right p.p. ring and $E(Q) = E(R)$. For any element a in R, $r_Q(a) = eQ$ for some $e \in E(Q) = E(R)$, and so $r_R(a) = r_Q(a) \cap R = eQ \cap R = eR$.

Proposition 2. *Let R be a right and left order in a ring Q. Then R is a right p.p. ring if and only if Q is a right p.p. ring and for each $e \in E(Q)$ there exists $f \in E(R)$ such that $eQ = fQ$.*

Proof. The proof of the if part is quite similar to that of Corollary 1, so we prove the only if part. Suppose that R is a right p.p. ring. By Proposition 1, Q is a right p.p. ring. For an arbitrary element e of $E(Q)$, we can write $1-e = a^{-1}b$ with $a, b \in R$. By hypothesis there exists $f \in E(R)$ such that $r_R(b) = fR$. Then we have $eQ = r_Q(1-e) = r_Q(a^{-1}b) = fQ$.

By a *classical right quotient ring* of a ring R, we mean a ring $Q \supseteq R$

such that every non zero-divisor d of R has an inverse d^{-1} in Q, and every element of Q has the form ab^{-1} for some $a, b \in R$. If a classical right quotient ring Q of R exists, it is unique up to isomorphism over R. A ring R is called a *right Ore ring* if it has a classical right quotient ring. Similarly, we define a *classical left quotient ring* and a *left Ore ring.*

The next is a slight generalization of [1, Lemma 3.1], [3, Theorem 3.4] and [19, Theorem 1.3].

Theorem 1. *Let R be a right Ore ring with classical right quotient ring Q. Then the following are equivalent:*

1) *R is a normal p.p. ring.*
2) *Q is a strongly regular ring and $E(Q)=E(R)$.*

Proof. By Corollary 1, it suffices to prove that if R is a normal p.p. ring, then Q is a strongly regular ring. Let ab^{-1} be an arbitrary element of Q $(a, b \in R)$. By [2, Lemma 2], there exist $e \in E(R)$ and a non zero-divisor $d \in R$ such that $a=ed$. Then we have $(ab^{-1})^2bd^{-1}=ab^{-1}e=aeb^{-1}=ab^{-1}$, and so Q is strongly regular.

Corollary 2. *The following are equivalent:*

1) *R is a normal p.p. ring and every non zero-divisor of R is invertible.*
2) *R is a strongly regular ring.*

Since a right duo ring is a normal right Ore ring, we readily obtain

Corollary 3. *Let R be a right duo p.p. ring with classical right quotient ring Q. Then Q is strongly regular and $E(Q)=E(R)$.*

Although a normal p.p. ring need not be a right Ore ring, we have

Theorem 2. *Let R be a normal p.p. ring. If R is integral over its center, then R has a strongly regular classical two-sided quotient ring.*

Proof. Clearly, the center of R is also a p.p. ring and every non zero-divisor of the center of R is a non zero-divisor in R. Therefore the classical quotient ring C of the center of R is a von Neumann regular ring (Theorem 1) and we may view R as a subring of the ring of central quotients Q of R. (cf. [15]) As is easily seen, Q is reduced and is integral over C, and hence by [17, Theorem 12] Q is also a strongly regular ring. Every non zero-divisor of R is a non zero-divisor in Q, and so it is invertible in Q. Hence Q is a classical two-sided quotient ring of R.

Using the standard technique, we can show that a ring which is a finitely generated module over its center is integral over the center, and hence we obtain

Corollary 4. *Let R be a normal p.p. ring. If R is a finitely generated module over its center, then R has a strongly regular classical two-sided quotient ring.*

According to Kaplansky [11], a ring R is called a *Baer ring* if every annihilator right ideal of R is generated by an idempotent, or equivalently, every annihilator left ideal of R is generated by an idempotent.

Let R be a right non-singular ring with maximal right quotient ring Q. Then R is said to be a *right Utumi ring* if it satisfies the following equivalent conditions (cf. [18, Theorem 2.2] or [16, XII, Proposition 4.7]):

1) Every non-essential right ideal of R has non-zero left annihilator.
2) Every closed right ideal of R is a right annihilator.
3) Every non-zero left ideal of Q has non-zero intersection with R.

The following lemma is available to simplify the proof of Utumi's theorem [18, Theorem 3.3]: Let R be a right and left non-singular ring. Then the maximal right and left quotient rings of R coincide if and only if R is a right and left Utumi ring.

Lemma 2. *Let R be a right non-singular ring with maximal right quotient ring Q. Then R is a right Utumi ring if and only if Q is a left quotient ring of R, that is, ${}_RQ$ is a rational extension of ${}_RR$.*

Proof. The if part is clear, so we prove the only if part. Since Q is a von Neumann regular ring, it suffices to show that $Qe \cap R \neq 0$ for every non-zero $e \in E(Q)$. First, we prove that $Qe = l_Q(R \cap (1-e)Q)$ for every $e \in E(Q)$. Let a be an arbitrary element of $l_Q(R \cap (1-e)Q)$. If we take an essential right ideal I of R such that $(1-e)I \subseteq R$, then $a(1-e)I=0$. Hence $a=ae$, which shows the above equality. If e is a non-zero idempotent of Q, then $R \cap (1-e)Q$ is a non-essential right ideal of R, hence by hypothesis $l_R(R \cap (1-e)Q) \neq 0$. Since $Qe = l_Q(R \cap (1-e)Q) \supseteq l_R(R \cap (1-e)Q) \neq 0$, we obtain $Qe \cap R \neq 0$.

Theorem 3. *Let R be a right non-singular ring with maximal right quotient ring Q. Then the following are equivalent:*

1) *R is a right Utumi, Baer ring.*

2) *For every* $e\in E(Q)$ *there exists* $f\in E(R)$ *such that* $Qe=Qf$.

If moreover R is normal, then 2) *is equivalent to the following condition*:

2′) $E(Q)=E(R)$.

Proof. 1)$\Rightarrow$ 2). For any non-zero $e\in E(Q)$, $R\cap(1-e)Q$ is a non-essential right ideal of R. Hence by hypothesis $l_R(R\cap(1-e)Q)=Rf$ for some non-zero $f\in E(R)$. On the other hand, since $l_Q(R\cap(1-e)Q)=Qe$ as was shown in the proof of Lemma 2, we have

$$Qe\cap R=l_R(R\cap(1-e)Q)=Rf=Qf\cap R.$$

Since R is a right Utumi ring, Q is a left quotient ring of R by Lemma 2. Hence Qe and Qf are essential extensions of $Qe\cap R(=Qf\cap R)$. Since $Z(_RQ)=0$ and Qe, Qf are closed submodules of $_RQ$, it follows from [5, §7, 7. Proposition] that $Qe=Qf$.

2)$\Rightarrow$ 1). As is clear from the assumption, Q is a left quotient ring of R. Hence, by Lemma 2, R is a right Utumi ring. Since every annihilator right ideal of Q is a closed right ideal [5, §8, 5. Proposition (3)], it is generated by an idempotent [5, §8, 4. Theorem (3)]. Therefore Q is a Baer ring. Hence for any non-empty subset X of R there exists $e\in E(Q)$ such that $l_Q(X)=Qe$. By hypothesis there exists $f\in E(R)$ such that $Qe=Qf$. Then we have

$$l_R(X)=l_Q(X)\cap R=Qe\cap R=Qf\cap R=Rf,$$

which shows that R is a Baer ring.

Thus we have proved the equivalence of 1) and 2). Noting that the centralizer of R in Q coincides with the center of Q, the last assertion is obvious.

In [14], Mewborn defined the Baer hull of a commutative semiprime ring. We now define the Baer hull of a reduced right Utumi ring.

Theorem 4. *Let R be a reduced right Utumi ring, and Q its maximal right quotient ring. Let* $B(R)$ *be the intersection of all Baer subrings of Q containing R. Then* $B(R)$ *is a Baer ring and coincides with the subring of Q generated by R and* $E(Q)$.

Proof. Let S be a subring of Q containing R. Then, by Theorem 3, S is a Baer ring if and only if $E(Q)=E(S)$. Therefore our assertion is clear.

The ring $B(R)$ defined in Theorem 4 will be called the *Baer hull* of the reduced right Utumi ring R. Suppose now that R is a reduced, right Utumi ring. Then the maximal right quotient ring Q of R is strongly regular ([16, XII, Proposition 5.2]). Hence the intersection $S(R)$ of all von Neumann regular subrings of Q containing R is strongly regular ([16, Exercise I. 47 (iii)]). We call $S(R)$ the *strongly regular hull* of R.

According to [13, Theorems 1 and 5], any semiprime PI-ring is a right (and left) Utumi ring. Hence we have

Corollary 5. *Every reduced PI-ring has the Baer hull and the strongly regular hull.*

Let R be a right Utumi, normal p.p. ring. If R has a classical right (or left) quotient ring Q, then $S(R)=Q$, and hence by Theorem 1 $E(S(R)) =E(R)$. We have the following conjecture.

Conjecture. If R is a right Utumi, normal p.p. ring, then $E(S(R))=E(R)$.

Let R be a ring and let M be a right R-module. For any $u \in M$ and right ideal I of R, we set $(I:u)=\{r \in R \mid ur \in I\}$.

Proposition 3 (cf. [14, Proposition 2.9]). *Let R be a reduced, right Utumi ring, and $S \supseteq R$ a normal, Baer ring such that S_R is non-singular. Then there is an isomorphism over R of $B(R)$ onto a subring of S.*

Proof. Let $S'=\{a \in S \mid (R:a)$ is an essential right ideal of $R\}$, and let K be a non-empty subset of S'. As is easily seen, S' is a subring of S containing R. Since S is a Baer ring, $r_S(K)=eS$ for some $e \in E(S)$. We show that $1-e \in S'$. Let I be a non-zero right ideal of R. If $KI=0$, then $I \subseteq eS$ and hence $(1-e)I=0$. Therefore, $(R:(1-e)) \supseteq I$. Thus, we may assume that there exist some $a \in K$ and $b \in I$ such that $ab \neq 0$. Clearly, $ab \in S'$. Since S_R is non-singular, there is an element $r \in R$ such that $0 \neq abr \in R$. Since R is reduced and S is normal, we have $0 \neq brabr \in (R:(1-e)) \cap I$. Therefore $(R:(1-e))$ is an essential right ideal of R. Hence $1-e \in S'$. Since S' is a subring of S we have $e \in S'$. Consequently, $r_{S'}(K)=r_S(K) \cap S'=eS \cap S'=eS'$. This shows that S' is a Baer ring. Since S' is a right quotient ring of R, there is a monomorphism over R of S' into the maximal right quotient ring of R. The rest of the proof is easy.

Let R be a ring and let M be a right R-module. We set $T(M)= \{m \in M \mid md=0$ for some non zero-divisor d of $R\}$. It is well known that

$T(M)$ is a submodule of M for each right R-module M if and only if R is a right Ore ring ([12, Theorem 1.4)). We say that M is *torsion free* if $T(M)=0$.

Lemma 3. *Let R be a ring and suppose that R has a classical right quotient ring Q.*

(1) *The following are equivalent:*

1) *Every torsion free right R-module is non-singular.*

2) *For every right R-module M we have $Z(M)=T(M)$.*

3) *Q is a semisimple Artinian ring.*

(2) *If every torsion free right R-module is a c.p. module, then Q is semisimple Artinian.*

Proof. (1) 1)⇒2). Let M be a right R-module and let m be an arbitrary element of $T(M)$. Then there exists a non zero-divisor d of R such that $md=0$. As is easily seen, dR is an essential right ideal of R, and so $m\in Z(M)$. Hence we have $T(M)\subseteq Z(M)$.

Now $M/T(M)$ is torsion free and hence by hypothesis it is non-singular. Since $Z(M)/T(M)\subseteq Z(M/T(M))$, it follows that $Z(M)=T(M)$.

2)⇒3). We show that Q has no proper essential right ideals. Let I be an arbitrary essential right ideal of Q. Considering Q/I as a right R-module, we have $T(Q/I)=Z(Q/I)=Q/I$. Hence there exists a non zero-divisor d of R such that $(1+I)d=I$, that is, $d\in I$. Hence we have $I=Q$.

3)⇒1). Let M be a torsion free right R-module. By [12, Proposition 1.5], M is an R-submodule of some Q-module. Let m be an arbitrary element of $Z(M)$. Then $r_R(m)$ contains a non zero-divisor d. Since d is invertible in Q, we get $m=0$. Thus we have proved that $Z(M)=0$.

(2) It is easily seen that every c.p. module is non-singular. Hence the assertion is a direct consequence of (1).

Theorem 5. *For a ring R the following are equivalent:*

1) *R has a classical two-sided quotient ring and every torsion free right R-module is a c.p. module.*

2) *R is a finite direct sum of prime right and left Goldie right p.p. rings.*

Proof. 1)⇒2). Let Q be a classical two-sided quotient ring of R. By Lemma 3, Q is semisimple Artinian. Clearly, R_R is torsion free and hence by hypothesis R is a right p.p. ring. Therefore, by [5, 20.32 Theorem], R is a finite direct sum of prime right and left Goldie right p.p. rings.

2) ⇒ 1). By hypothesis R has a classical two-sided quotient ring Q which is semisimple Artinian. Let M be a torsion free right R-module. By [12, Proposition 1.5], M is an R-submodule of some Q-module. Let m be an arbitrary element of M. Since Q is semisimple Artinian, $r_Q(m)=eQ$ for some $e\in E(Q)$. Noting that R is a right p.p. ring, there exists $f\in E(R)$ such that $eQ=fQ$ (Proposition 2), and hence $r_R(m)=r_Q(m)\cap R=fQ\cap R=fR$. This completes the proof.

Theorem 6 (cf. [3, Proposition 4.8]). *Let R be a normal, right Ore ring with classical right quotient ring Q. Then the following are equivalent:*

1) *Every torsion free right R-module is a c.p. module.*
2) *R is a p.p. ring and Q is a finite direct sum of division rings.*
3) *R is a finite direct sum of right Ore domains.*

Proof. 1) ⇒ 2). Clearly R is a p.p. ring and, by Lemma 3 (2), Q is semisimple Artinian. Noting that Q is normal (Theorem 1), Q is a finite direct sum of division rings.

2) ⇒ 3). By Theorem 1, $E(R)=E(Q)$, and hence this implication is clear.

3) ⇒ 1). Since $E(Q)=E(R)$, the proof is similar to that of 2) ⇒ 1) in Theorem 5.

A right R-module M is said to be *divisible* if $Md=M$ for every non zero-divisor d of R, and M is called *p-injective* if for any principal right ideal I of R and any R-homomorphism $f: I\to M$, there exists $m\in M$ such that $f(a)=ma$ for all $a\in I$.

Theorem 7 (cf. [12, Theorem 3.4]). *Let R be a ring and suppose that R has a semisimple Artinian classical two-sided quotient ring. Then the following are equivalent:*

1) *Every divisible right R-module is p-injective.*
2) *R is a right p.p. ring.*

Proof. 1) ⇒ 2). Let M be a p-injective right R-module. Then, by the proof of [12, Theorem 3.1], M is divisible. If $\bar{M}$ is an arbitrary factor module of M, then $\bar{M}$ is also divisible, and hence by hypothesis it is p-injective. Therefore, by [20, Remarks (2)], R is a right p.p. ring.

2) ⇒ 1). Let M be a divisible right R-module and let f be an R-homomorphism of a principal right ideal I of R to M. Since R is of finite right Goldie dimension, there exist principal right ideals $I_1, \cdots, I_n$ such that

$J=I\oplus I_1\oplus\cdots\oplus I_n$ is an essential right ideal of R. Since R is a right p.p. ring, J is projective. By [8, Theorem 3.9], J contains a non zero-divisor. Hence, in view of [12, Lemma 3.8] we can apply the argument used in the proof of (2)⇒(1) in [12, Theorem 3.4] to conclude that f can be extended to an element of $\mathrm{Hom}_R(R,M)$. This completes the proof.

Remark. Suppose that R has a classical right quotient ring Q. Then every torsion free right R-module is p-injective if and only if Q is von Neumann regular.

It is well known that a module M over a domain R is p-injective if and only if it is divisible (see e.g. [10, Lemma 2]).

Theorem 8. *If R is a normal ring, then the following are equivalent:*
1) *Every divisible right R-module is p-injective.*
2) *R is a p.p. ring.*

Proof. 1)⇒2). The proof is same as that of 1)⇒2) in Theorem 7.
2)⇒1). Let M be a divisible right R-module, and let $f: aR\to M$ $(a\in R)$ be an R-homomorphism. Since R is a right p.p. ring, $r_R(a)=eR$ for some $e\in E(R)$. Noting that $a+e$ is a non zero-divisor, we see that there exists $m\in M$ such that $m(a+e)=f(a)$. Since $f(a)=f(a(1-e))=f(a)(1-e)$, we have then $f(a)=ma$. This shows that M is p-injective.

A ring R is called a *completely right p.p. ring* (resp. *completely hereditary ring*) if every non-zero factor ring of R is right p.p. (resp. right hereditary).

Lemma 4. *If R is a completely right p.p. ring, then the center C of R is a von Neumann regular ring.*

Proof. First, we show that C is semiprime. Let a be an element of C with $a^2=0$. Since R is right p.p., $r_R(a)=eR$ for some $e\in E(R)$. Then we have $a\in eR\cap R(1-e)\cap C=0$, and therefore C is semiprime. Next, let c be an arbitrary element of C and suppose that $c^2R\neq R$. Since the center of R/c^2R is also semiprime, we obtain $c\in c^2R$. Now it is easy to see that C is a von Neumann regular ring.

Corollary 6. *Let R be a normal, completely right p.p. ring satisfying a polynomial identity. Then R is a strongly regular ring.*

Proof. By Lemma 4, the center of each prime factor ring of R is a field. Hence, by [15, Corollary 1.6.28], each prime factor ring of R is simple Artinian. Therefore, by [6, Corollary 1.4], R is a strongly regular ring.

Theorem 9. *Let R be a completely right p.p. ring satisfying a polynomial identity. If R is right Noetherian, then R is right Artinian.*

Proof. In the proof of Corollary 6, we have seen that each prime factor ring of R is a von Neumann regular ring. Hence, by [6, Corollary 1.2], $R/P(R)$ is von Neumann regular and $P(R)$ is nilpotent, where $P(R)$ is the prime radical of R. Since R is right Noetherian, $R/P(R)$ is semisimple Artinian. Therefore R is right Artinian.

Combining Theorem 9 with [7, (4.2) Theorem], we readily obtain

Corollary 7. *If R is a PI-ring, then the following are equivalent:*

1) *R is a completely hereditary right Noetherian ring.*

2) *R is a hereditary, right Artinian ring and the square of its radical is* 0.

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Lemma 2 in [2] is false. In fact, let R be a right Ore domain which is not left Ore (see e.g., [3, Example 1.3.7]). Let Q denote the skew field of right fractions of R. (Obviously Q is a maximal right quotient ring of R.) Then R is a right Utumi ring, but Q is not a left quotient ring of R. Lemma 2 was used in the proof of Theorem 3. We present a revised version of Theorem 3. The other results of the paper remain true without change.

Theorem 3'. *Let R be a right Utumi ring with maximal right quotient ring Q. Then the following are equivalent:*

1) *R is a Baer ring.*

2) *For every $e \in E(Q)$ there exists $f \in E(R)$ such that $Qe \cap R = Rf$.*

If moreover R is normal, then 2) is equivalent to the following condition:

2') $E(Q) = E(R)$.

Proof. 1) ⇒ 2). We claim that $Qe = l_Q(R \cap (1-e)Q)$ for every $e \in E(Q)$. Indeed, let a be an arbitrary element of $l_Q(R \cap (1-e)Q)$. If we take an essential right ideal I of R such that $(1-e)I \subseteq R$, then $a(1-e)I = 0$. Hence $a = ae$, which shows the above equality. Now for any non-zero $e \in E(Q)$, $R \cap (1-e)Q$ is a non-essential right ideal of R. Hence by hypothesis $l_R(R \cap (1-e)Q) = Rf$ for some non-zero $f \in E(R)$. Combining this with what we have shown above, we have $Qe \cap R = l_R(R \cap (1-e)R) = Rf$.

2) ⇒ 1). Since every annihilator right ideal of Q is a closed right ideal [1, Proposition 8.5 (3)], it is generated by an idempotent [1, Theorem 8.4 (3)]. Therefore Q is a Baer ring. Hence for any non-empty subset X of R there exists $e \in E(Q)$ such that $l_Q(X) = Qe$. By hypothesis there exists $f \in E(R)$ such that $Qe \cap R = Rf$, and so $l_R(X) = l_Q(X) \cap R = Qe \cap R = Rf$. This proves that R is a Baer ring.

Thus we have proved the equivalence of 1) and 2). Trivially 2') implies 2). Now suppose R is normal and 2) holds. Since the centralizer of R in Q coincides with the center of Q, every idempotent of R is central in Q. Let $e \in E(Q)$ and take $f \in E(R)$ such that $Qe \cap R = Rf$. Clearly we have $f = fe$ and $Q(1-f)e \cap R = 0$. Since R is right Utumi, the latter implies $(1-f)e = 0$. Therefore we have $f = fe = e$. This completes the proof.

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