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SUBGROUP $SU(8)/Z_2$ OF COMPACT SIMPLE LIE GROUP E_7 AND NON-COMPACT SIMPLE LIE GROUP $E_{7(7)}$ OF TYPE E_7

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It is known that there exist four simple Lie groups of type E_7 up to local isomorphism, one of them is compact and the others are non-compact. We have shown that in [3], [4] the group

$$E_7 = \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \mid \alpha(P\times Q)\alpha^{-1} = \alpha P\times \alpha Q, \langle \alpha P, \alpha Q\rangle = \langle P, Q\rangle\}$$

is a simply connected compact simple Lie group of type E_7 and in [5], [7] the groups

$$E_{7(-25)} = \{\alpha \in \mathrm{Iso}_R(\mathfrak{P}, \mathfrak{P}) \mid \alpha(P\times Q)\alpha^{-1} = \alpha P\times \alpha Q\},$$
$$E_{7(-5)} = \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \mid \alpha(P\times Q)\alpha^{-1} = \alpha P\times \alpha Q, \langle \alpha P, \alpha Q\rangle_\sigma = \langle P, Q\rangle_\sigma\}$$

are connected non-compact simple Lie groups of type $E_{7(-25)}$, $E_{7(-5)}$ respectively, and their polar decompositions are given by

$$E_{7(-25)} \simeq (U(1)\times E_6)/\boldsymbol{Z}_3\times \boldsymbol{R}^{54},$$
$$E_{7(-5)} \simeq (SU(2)\times Spin(12))/\boldsymbol{Z}_2\times \boldsymbol{R}^{64}.$$

In this paper, first we find a subgroup in E_7 which is isomorphic to the group $SU(8)/\boldsymbol{Z}_2$. Next we show that the group

$$E_{7(7)} = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P}', \mathfrak{P}') \mid \alpha(P\times Q)\alpha^{-1} = \alpha P\times \alpha Q\}$$

is a connected non-compact simple Lie group of type $E_{7(7)}$ with the center $z(E_{7(7)}) = \{1, -1\}$ and its polar decomposition is given by

$$E_{7(7)} \simeq SU(8)/\boldsymbol{Z}_2\times \boldsymbol{R}^{70}.$$

Our main method used in this paper is to give homomorphisms $\psi : SU(8) \to E_7$ and $\psi : SU(8)\to E_{7(7)}$ explicitly.

I. Subgroup $SU(8)/Z_2$ of compact simple Lie group E_7

1. Preliminaries

1.1. Cayley algebras $\mathfrak{C}$, $\mathfrak{C}^C$ and exceptional Jordan algebras $\mathfrak{J}$, $\mathfrak{J}^C$. Let $\mathfrak{C} = \boldsymbol{H} \oplus \boldsymbol{H}e$ ($\boldsymbol{H}$ is the quaternionic field) denote the Cayley algebra over the field $\boldsymbol{R}$ of real numbers with the multiplication

$$(a+be)(c+de) = (ac-\bar{d}b)+(b\bar{c}+da)e$$

and $\mathfrak{C}^C = \{x_1+ix_2 \mid x_1, x_2 \in \mathfrak{C}\}$ its complexification with respect to the field $\boldsymbol{C}$ of complex numbers.

Let $\mathfrak{J} = \{X \in M(3, \mathfrak{C}) \mid X^* = X\} = \left\{ \begin{bmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{bmatrix} \middle| \; \xi_i \in \boldsymbol{R}, \; x_i \in \mathfrak{C} \right\}$ denote the exceptional Jordan algebra with the multiplication

$$X \circ Y = (XY+YX)/2$$

and $\mathfrak{J}^C = \{X_1+iX_2 \mid X_1, X_2 \in \mathfrak{J}\}$ its complexification. In $\mathfrak{J}$ and $\mathfrak{J}^C$, inner products (X, Y), $\langle X, Y\rangle$, the crossed product $X\times Y$, the trilinear form (X, Y, Z) and the determinant $\det X$ are defined respectively by

$$(X, Y) = \mathrm{tr}(X\circ Y), \qquad \langle X, Y\rangle = (\tau X, Y) = (\bar{X}, Y),$$
$$X\times Y = (2X\circ Y-\mathrm{tr}(X)Y-\mathrm{tr}(Y)X+(\mathrm{tr}(X)\mathrm{tr}(Y)-\mathrm{tr}(X\circ Y))E),$$
$$(X, Y, Z) = (X, Y\times Z), \qquad \det X = (X, X, X)/3$$

where $\tau : \mathfrak{J}^C \to \mathfrak{J}^C$ denotes the complex conjugation : $\tau(X_1+iX_2) = X_1-iX_2$, $X_1, X_2 \in \mathfrak{J}$ (τX is also denoted by $\bar{X}$) and E the 3×3 unit matrix. (The other $n\times n$ unit matrix will be also denoted by E).

In $\mathfrak{J}$ and $\mathfrak{J}^C$ we adopt the following notations :

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$F_1(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{bmatrix}, \quad F_2(x) = \begin{bmatrix} 0 & 0 & \bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{bmatrix}, \quad F_3(x) = \begin{bmatrix} 0 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the table of the crossed products among them are given by

$$\begin{cases} E_i\times E_i = 0, & E_i\times E_{i+1} = E_{i+2}/2, \\ E_i\times F_i(x) = -F_i(x)/2, & E_i\times F_j(x) = 0, \quad i \neq j, \\ F_i(x)\times F_i(y) = -(x, y)E_i, & F_i(x)\times F_{i+1}(y) = F_{i+2}(\overline{xy})/2 \end{cases}$$

where indexes are considered as mod 3.

Finally we define a linear involution $\gamma : \mathfrak{J} \to \mathfrak{J}$ (resp. $\mathfrak{J}^C \to \mathfrak{J}^C$) by

$$\gamma\begin{bmatrix} \xi_1 & a_3+b_3e & \bar{a}_2-b_2e \\ \bar{a}_3-b_3e & \xi_2 & a_1+b_1e \\ a_2+b_2e & \bar{a}_1-b_1e & \xi_3 \end{bmatrix} = \begin{bmatrix} \xi_1 & a_3-b_3e & \bar{a}_2+b_2e \\ \bar{a}_3+b_3e & \xi_2 & a_1-b_1e \\ a_2-b_2e & \bar{a}_1+b_1e & \xi_3 \end{bmatrix}$$

where $\xi_i \in \boldsymbol{R}$ (resp. $\boldsymbol{C}$), $a_i, b_i \in \boldsymbol{H}$ (resp. $\boldsymbol{H}^C$ (the complexification of $\boldsymbol{H}$)).

1.2. Compact Lie group E_6 and subgroup $(E_6)_\rho$. We have shown in [8] that the group

$$\begin{aligned} E_6 &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C, \mathfrak{J}^C) \mid \det\alpha X = \det X, \langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle\} \\ &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{J}^C, \mathfrak{J}^C) \mid \tau\alpha\tau(X\times Y) = \alpha X\times\alpha Y, \langle \alpha X, \alpha Y\rangle = \langle X, Y\rangle\} \end{aligned}$$

is a simply connected compact simple Lie group of type E_6 and therefore its Lie algebra

$$\mathfrak{e}_6 = \{\phi \in \mathrm{Hom}_C(\mathfrak{J}^C, \mathfrak{J}^C) \mid (\phi X, X, X) = 0, \langle \phi X, Y\rangle = -\langle X, \phi Y\rangle\}$$

is a compact simple Lie algebra of type E_6.

We have also found in [8] a subgroup of type C_4 in the group E_6. For later use we review this subgroup. Let $\mathfrak{J}(4, \boldsymbol{H}) = \{X \in M(4, \boldsymbol{H}) \mid X^* = X\}$ denote the vector space of all 4×4 quaternionic Hermitian matrices and $\mathfrak{J}(4, \boldsymbol{H})^C = \{X_1 + iX_2 \mid X_1, X_2 \in \mathfrak{J}(4, \boldsymbol{H})\}$ its complexification. In $\mathfrak{J}(4, \boldsymbol{H})$ and $\mathfrak{J}(4, \boldsymbol{H})^C$, Jordan multiplications $X\circ Y$ are defined by $X\circ Y = (XY+YX)/2$. Put $\mathfrak{J}(4, \boldsymbol{H})_0 = \{X \in \mathfrak{J}(4, \boldsymbol{H}) \mid \mathrm{tr}(X) = 0\}$ and let $\mathfrak{J}(4, \boldsymbol{H})_0{}^C = \{X \in \mathfrak{J}(4, \boldsymbol{H})^C \mid \mathrm{tr}(X) = 0\} = \{X_1 + iX_2 \mid X_1, X_2 \in \mathfrak{J}(4, \boldsymbol{H})_0\}$ be the complexification of $\mathfrak{J}(4, \boldsymbol{H})_0$. Now, we define a mapping $g : \mathfrak{J}^C \to \mathfrak{J}(4, \boldsymbol{H})_0{}^C$ by

$$g\left(\begin{bmatrix} \xi_1 & a_3+b_3e & \bar{a}_2-b_2e \\ \bar{a}_3-b_3e & \xi_2 & a_1+b_1e \\ a_2+b_2e & \bar{a}_1-b_1e & \xi_3 \end{bmatrix} + i\begin{bmatrix} \eta_1 & c_3+d_3e & \bar{c}_2-d_2e \\ \bar{c}_3-d_3e & \eta_2 & c_1+d_1e \\ c_2+d_2e & \bar{c}-d_1e & \eta_3 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \lambda_1 & -d_1 & -d_2 & -d_3 \\ -\bar{d}_1 & \lambda_2 & a_3 & \bar{a}_2 \\ -\bar{d}_2 & \bar{a}_3 & \lambda_3 & a_1 \\ -\bar{d}_3 & a_2 & \bar{a}_2 & \lambda_4 \end{bmatrix} + i\begin{bmatrix} \mu_1 & b_1 & b_2 & b_2 \\ \bar{b}_1 & \mu_2 & c_3 & \bar{c}_2 \\ \bar{b}_2 & \bar{c}_3 & \mu_3 & c_1 \\ \bar{b}_3 & c_2 & \bar{c}_1 & \mu_4 \end{bmatrix}$$

where $\xi_i, \eta_i \in \boldsymbol{R}$, $a_i, b_i, c_i, d_i \in \boldsymbol{H}$ and $\lambda_1 = (\xi_1+\xi_2+\xi_3)/2$, $\lambda_2 = (\xi_1-\xi_2-\xi_3)/2$, $\lambda_3 = (\xi_2-\xi_1-\xi_3)/2$, $\lambda_4 = (\xi_3-\xi_1-\xi_2)/2$, $\mu_1 = (\eta_1+\eta_2+\eta_3)/2$, $\mu_2 = (\eta_1-\eta_2-\eta_3)/2$, $\mu_3 = (\eta_2-\eta_1-\eta_3)/2$, $\mu_4 = (\eta_3-\eta_1-\eta_2)/2$. And we define a conjugate linear involutive transformation of $\mathfrak{J}^C$ by

$$\rho = \tau\gamma = \gamma\tau.$$

Lemma 1 (1) ([8, Lemma 17]). *The mapping $g : \mathfrak{J}^C \to \mathfrak{J}(4, \boldsymbol{H})_0{}^C$ is a $\boldsymbol{C}$-isomorphism satisfying*

$$g(X\times Y) = g(\gamma X)\circ g(\gamma Y) - ((\gamma X, Y)/4)E$$

where E is the 4×4 unit matrix.

(2) *Put $(\mathfrak{J}^C)_\rho = \{X \in \mathfrak{J}^C \mid \rho X = X\}$. Then g induces an $\boldsymbol{R}$-isomorphism*

$$g : (\mathfrak{J}^C)_\rho \to \mathfrak{J}(4, \boldsymbol{H})_0.$$

We have shown in [8, Theorem 18] that a subgroup $(E_6)_\rho$ of E_6

$$(E_6)_\rho = \{\alpha \in E_6 \mid \rho\alpha = \alpha\rho\}$$

is isomorphic to the group $Sp(4)/\boldsymbol{Z}_2$ (where $Sp(4) = \{A \in M(4, \boldsymbol{H}) \mid A^*A = E\}$ is the symplectic group and $\boldsymbol{Z}_2 = \{E, -E\}$) by the correspondence

$$\varphi : Sp(4) \to (E_6)_\rho, \quad \varphi(C)X = g^{-1}(C(gX)C^*), \quad X \in \mathfrak{J}^C$$

with $\mathrm{Ker}\varphi = \boldsymbol{Z}_2$. Therefore its Lie algebra

$$(\mathfrak{e}_6)_\rho = \{\phi \in \mathfrak{e}_6 \mid \rho\phi = \phi\rho\}$$

is isomorphic to the symplectic Lie algebra $\mathfrak{sp}(4) = \{C \in M(4, \boldsymbol{H}) \mid C^* = -C\}$ by the correspondence

$$d\varphi : \mathfrak{sp}(4) \to (\mathfrak{e}_6)_\rho, \quad d\varphi(C)X = g^{-1}(C(gX) - (gX)C), \quad X \in \mathfrak{J}^C.$$

Finally, we note that the complexification Lie algebra ${\mathfrak{e}_6}^C$ of $\mathfrak{e}_6$:

$${\mathfrak{e}_6}^C = \{\phi \in \mathrm{Hom}_C(\mathfrak{J}^C, \mathfrak{J}^C) \mid (\phi X, X, X) = 0\}$$

is a simple Lie algebra over $\boldsymbol{C}$ of type E_6. And, for $A, B \in \mathfrak{J}^C$, $A \vee B \in {\mathfrak{e}_6}^C$ is defined by

$$(A \vee B)X = ((B, X)/2)A + ((A, B)/6)X - 2B \times (A \times X), \quad X \in \mathfrak{J}^C.$$

1.3. Compact Lie group E_7 and its Lie algebra $\mathfrak{e}_7$. Let $\mathfrak{P}^C$ be a 56 dimensional vector space over $\boldsymbol{C}$ defined by

$$\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus \boldsymbol{C} \oplus \boldsymbol{C}.$$

In $\mathfrak{P}^C$, the positive definite inner product $\langle P, Q\rangle$ and the skew-symmetric inner product $\{P, Q\}$ are defined respectively by

$$\langle P, Q\rangle = \langle X, Z\rangle + \langle Y, W\rangle + \bar{\xi}\zeta + \bar{\eta}\omega,$$
$$\{P, Q\} = (X, W) - (Y, Z) + \xi\omega - \eta\zeta$$

for $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}^C$.

For $\phi \in {\mathfrak{e}_6}^C$, $A, B \in \mathfrak{J}^C$ and $\nu \in \boldsymbol{C}$, we define a linear transformation $\Phi(\phi, A, B, \nu)$ of $\mathfrak{P}^C$ by

$$\Phi(\phi, A, B, \nu)\begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \phi - (\nu/3)1 & 2B & 0 & A \\ 2A & \phi' + (\nu/3)1 & B & 0 \\ 0 & A & \nu & 0 \\ B & 0 & 0 & -\nu \end{bmatrix}\begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix}$$
$$= \begin{bmatrix} \phi X - (\nu/3)X + 2B \times Y + \eta A \\ 2A \times X + \phi' Y + (\nu/3)Y + \xi B \\ (A, Y) + \nu\xi \\ (B, X) - \nu\eta \end{bmatrix}$$

where $\phi' \in \mathfrak{e}_6{}^C$ denotes the skew-transpose of ϕ with respect to the inner product (X, Y) : $(\phi X, Y)+(X, \phi' Y)=0$. For $P=(X, Y, \xi, \eta)$, $Q=(Z, W, \zeta, \omega) \in \mathfrak{P}^C$, we define a linear transformation $P \times Q$ of $\mathfrak{P}^C$ by

$$P \times Q = \Phi(\phi, A, B, \nu), \begin{cases} \phi = -(X \vee W + Z \vee Y)/2, \\ A = -(2Y \times W - \xi Z - \zeta X)/4, \\ B = (2X \times Z - \eta W - \omega Y)/4, \\ \nu = ((X, W)+(Z, Y)-3(\xi\omega + \zeta\eta))/8. \end{cases}$$

And we define a submanifold $\mathfrak{M}^C$ of $\mathfrak{P}^C$, called Freudenthal manifold, by

$$\begin{aligned} \mathfrak{M}^C &= \{P \in \mathfrak{P}^C \mid P \times P = 0, \ P \neq 0\} \\ &= \{P=(X, Y, \xi, \eta) \in \mathfrak{P}^C \left| \begin{matrix} X \vee Y = 0, \ X \times X = \eta Y \\ Y \times Y = \xi X, \ (X, Y) = 3\xi\eta, \end{matrix} \right. P \neq 0\}. \end{aligned}$$

Now, as stated in the introduction, a simply connected compact simple Lie group of type E_7 is given by

$$\begin{aligned} E_7 &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\} \\ &= \{\alpha \in \mathrm{Iso}_C(\mathfrak{P}^C, \mathfrak{P}^C) \mid \alpha \mathfrak{M}^C = \mathfrak{M}^C, \{\alpha P, \alpha Q\} = \{P, Q\}, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\} \end{aligned}$$

and its Lie algebra is

$$\mathfrak{e}_7 = \{\Phi(\phi, A, -\bar{A}, \nu) \in \mathrm{Hom}_C(\mathfrak{P}^C, \mathfrak{P}^C) \mid \phi \in \mathfrak{e}_6, \ A \in \mathfrak{J}^C, \ \nu \in C, \ \bar{\nu} = -\nu\}.$$

The group E_7 contains a subgroup

$$\tilde{E}_6 = \{\alpha \in E_7 \mid \alpha(0, 0, 1, 0) = (0, 0, 1, 0)\}$$

which is isomorphic to the group E_6 by the coorespondence

$$E_6 \ni \alpha \longleftrightarrow \alpha = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \tau\alpha\tau & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \tilde{E}_6 \subset E_7$$

([3] Proposition 2). From now on, we identify the group E_6 with the group $\tilde{E}_6$.

2. Subgroup $(E_7)_\rho$ of E_7 and its Lie algebra $(\mathfrak{e}_7)_\rho$

We define a conjugate linear involution ρ of $\mathfrak{P}^C$ (used the same notation ρ in $\mathfrak{J}^C$) by

$$\rho(X, Y, \xi, \eta) = (\rho X, \rho Y, \bar{\xi}, \bar{\eta})$$

and we shall investigate a subgroup $(E_7)_\rho$ of E_7

$$(E_7)_\rho = \{\alpha \in E_7 \mid \rho\alpha = \alpha\rho\}.$$

For this purpose, we give some preliminaries [8].

The quaternionic field $\boldsymbol{H} = \boldsymbol{C} \oplus j\boldsymbol{C}$ is isomorphic to the space $\mathfrak{H} = \{\boldsymbol{x} \in M(2, \boldsymbol{C}) \mid \boldsymbol{x}\boldsymbol{j} = \boldsymbol{j}\boldsymbol{x}\}$, where $\boldsymbol{j} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, as an algebra by the correspondence $k : \boldsymbol{H} \to \mathfrak{H}$,

$$k(a+jb) = \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \quad a, b \in \boldsymbol{C}.$$

This mapping k is naturally extended to the mappings

$$k : M(4, \boldsymbol{H}) \to M(8, \boldsymbol{C}), \qquad k : M(4, \boldsymbol{H})^C \to M(8, \boldsymbol{C})$$

$$k\Big(\; x_{st} \; \Big) = \Big(\; k(x_{st}) \; \Big), \quad k\Big(\; x_{st} + iy_{st} \; \Big) = \Big(\; k(x_{st}) + ik(y_{st}) \; \Big)$$

$x_{st}, y_{st} \in \boldsymbol{H}$, respectively. (In the latter equation, i in the left side is the complexification unit of $\mathfrak{J}(4, \boldsymbol{H})^C$ and i in the right side is the imaginary unit of the field $\boldsymbol{C}$).

Let $SU(8) = \{A \in M(8, \boldsymbol{C}) \mid A^*A = E, \det A = 1\}$ be the special unitary group. The symplectic group $Sp(4)$ is often regarded as a subgroup of $SU(8)$ by

$$Sp(4) = k(Sp(4)) = \{A \in SU(8) \mid AJ{}^tA = J\}, \qquad J = \begin{bmatrix} \boldsymbol{j} & 0 & 0 & 0 \\ 0 & \boldsymbol{j} & 0 & 0 \\ 0 & 0 & \boldsymbol{j} & 0 \\ 0 & 0 & 0 & \boldsymbol{j} \end{bmatrix}$$

Lemma 2. *Any element D of the special unitary Lie algebra $\mathfrak{su}(8) = \{D \in M(8, \boldsymbol{C}) \mid D^* = -D, \operatorname{tr}(D) = 0\}$ is represented by the form*

$$D = k(C) + ik(B) \qquad C \in \mathfrak{sp}(4), \ B \in \mathfrak{J}(4, \boldsymbol{H})_0.$$

Proof. For $D \in \mathfrak{su}(8)$, put $C_1 = (D - J\bar{D}J)/2$ and $B_1 = -i(D + J\bar{D}J)/2$, then

$$D = C_1 + iB_1 \quad C_1{}^* = -C_1, \ C_1J = J\bar{C}_1, \ B_1{}^* = B_1, \ B_1J = J\bar{B}_1, \ \operatorname{tr}(B_1) = 0.$$

So, $C = k^{-1}(C_1)$ and $B = k^{-1}(B_1)$ are required ones.

Proposition 3. *The Lie algebra $(\mathfrak{e}_7)_\rho$ of the group $(E_7)_\rho$ is*

$$\begin{aligned}(\mathfrak{e}_7)_\rho &= \{\Phi \in \mathfrak{e}_7 \mid \rho\Phi = \Phi\rho\} \\ &= \{\Phi_\rho(\phi, A) \in \mathfrak{e}_7 \mid \phi \in \mathfrak{e}_6, \ \rho\phi = \phi\rho, \ A \in \mathfrak{J}^C, \ \rho A = A\} \\ &= \{\Phi_\rho(d\varphi(C), g^{-1}B) \in \mathfrak{e}_7 \mid C \in \mathfrak{sp}(4), \ B \in \mathfrak{J}(4, \boldsymbol{H})_0\}.\end{aligned}$$

where $\Phi_\rho(\phi, A) = \Phi(\phi, A, -\gamma A, 0)$. The Lie bracket $[\Phi_1 \ \Phi_2]$ in $(\mathfrak{e}_7)_\rho$ is given by

$$[\Phi_\rho(\phi_1, A_1), \Phi_\rho(\phi_2, A_2)] = \Phi_\rho(\phi, A)$$

where

$$\phi = [\phi_1, \ \phi_2]-2A_1 \vee \bar{A}_2+2A_2 \vee \bar{A}_1, \qquad A = \phi_1 A_2-\phi_2 A_1.$$

And $(\mathfrak{e}_7)_\rho$ *is isomorphic to the special unitary Lie algebra* $\mathfrak{su}(8)$ *by the correspondence*

$$k(C)+ik(B) \in \mathfrak{su}(8) \xrightarrow{d\psi} \Phi_\rho(d\varphi(C), \ g^{-1}B) \in (\mathfrak{e}_7)_\rho$$

where $C \in \mathfrak{sp}(4)$, $B \in \mathfrak{J}(4, \ \boldsymbol{H})_0$.

Proof. The first statements are easily shown. We shall show that $d\psi : \mathfrak{su}(8) \to (\mathfrak{e}_7)_\rho$ is an isomorphism. (This is the direct consequence of the following section 4, however, here, we will give the direct proof).

(1) For $C_1, \ C_2 \in \mathfrak{sp}(4)$, $[k(C_1), \ k(C_2)] = k[C_1, \ C_2] \to \Phi_\rho(d\varphi[C_1, \ C_2], 0) = [\Phi_\rho(d\varphi(C_1), \ 0), \ \Phi_\rho(d\varphi(C_2), \ 0)]$.

(2) For $C \in \mathfrak{sp}(4)$, $B \in \mathfrak{J}(4, \ \boldsymbol{H})_0$, $[k(C), \ ik(B)] = ik[C, \ B] \to \Phi_\rho(0, g^{-1}[C, \ B]) = \Phi_\rho(0, \ d\varphi(C)(g^{-1}B)) = [\Phi_\rho(d\varphi(C), \ 0), \ \Phi_\rho(0, \ g^{-1}B)]$.

(3) For $B_1, \ B_2 \in \mathfrak{J}(4, \ \boldsymbol{H})_0$, $[ik(B_1), \ ik(B_2)] = -k[B_1, \ B_2] \to \Phi_\rho(-d\varphi[B_1, \ B_2], \ 0)$. On the other hand, (put $A_1 = g^{-1}B_1$, $A_2 = g^{-1}B_2$)

$$\begin{aligned}[\Phi_\rho(0, \ g^{-1}B_1), \ \Phi_\rho(0, \ g^{-1}B_2)] &= [\Phi_\rho(0, \ A_1), \ \Phi_\rho(0, \ A_2)] \\ &= \Phi_\rho(-2A_1 \vee \bar{A}_2+2A_2 \vee \bar{A}_1, \ 0)\end{aligned}$$

where $g((2A_1 \vee \bar{A}_2-2A_2 \vee \bar{A}_1)X) \qquad X \in \mathfrak{J}^C$

$$\begin{aligned}
&= g((2A_1 \vee \gamma A_2 -2A_2 \vee \gamma A_1)X) \\
&= g((\gamma A_2, \ X)A_1+((A_1, \ \gamma A_2)/3)X-4\gamma A_2\times(A_1\times X) \\
&\quad -((\gamma A_1, \ X)A_2+((A_2, \ \gamma A_1)/3)X-4\gamma A_1\times(A_2\times X))) \\
&= (\gamma A_2, \ X)gA_1+((A_1, \ \gamma A_2)/3)gX-4gA_2\circ g(\gamma A_1\times\gamma X)+(A_2, \ A_1\times X)E) \\
&\quad -((\gamma A_1, \ X)gA_2+((A_2, \ \gamma A_1)/3)gX-4gA_1\circ g(\gamma A_2\times\gamma X)+(A_1, \ A_2\times X)E)) \\
&= (\gamma A_2, \ X)B_1-4B_2\circ(B_1\circ gX-((\gamma A_1, \ X)/4)E) \\
&\quad -((\gamma A_1, \ X)B_2-4B_1\circ(B_2\circ gX-((\gamma A_2, \ X)/4)E)) \\
&= 4B_1\circ(B_2\circ gX)-4B_2\circ(B_1\circ gX) \\
&= B_1B_2(gX)+B_1(gX)B_2+B_2(gX)B_1+(gX)B_2B_1 \\
&\quad -(B_2B_1(gX)+B_2(gX)B_1+B_1(gX)B_2+(gX)B_1B_2) \\
&= [B_1, \ B_2](gX)-(gX)[B_1, \ B_2] = g((d\varphi[B_1, \ B_2])X).
\end{aligned}$$

Therefore we have $\Phi_\rho(-d\varphi[B_1, \ B_2], \ 0) = [\Phi_\rho(0, \ g^{-1}B_1), \ \Phi_\rho(0, \ g^{-1}B_2)]$. Thus Proposition 3 has been proved.

3. C-isomorphism χ between $\mathfrak{P}^C$ and $\mathfrak{S}(8, \ C) \oplus \mathfrak{S}(8, \ C)$

Let $\mathfrak{S}(8, \ \boldsymbol{C})$ denote the 28 dimensional vector space over $\boldsymbol{C}$ of all 8×8 complex skew-symmetric matrices:

$$\mathfrak{S}(8, \ \boldsymbol{C}) = \{S \in M(8, \ \boldsymbol{C}) \mid {}^tS = -S\}.$$

We give a C-isomorphism $\chi : \mathfrak{P}^C \to \mathfrak{S}(8, C) \oplus \mathfrak{S}(8, C)$ by

$$\chi = h\varepsilon\tilde{g}\gamma_2$$

where γ_2, $\tilde{g}$, ε, h are C-isomorphisms defined by

$\gamma_2 : \mathfrak{P}^C \to \mathfrak{P}^C$, $\gamma_2(X, Y, \xi, \eta) = (X, \gamma Y, \xi, \eta)$,
$\tilde{g} : \mathfrak{P}^C \to \mathfrak{J}(4, \boldsymbol{H})^C \oplus \mathfrak{J}(4, \boldsymbol{H})^C$,
$\tilde{g}(X, Y, \xi, \eta) = (gX-(\xi/2)E, gY-(\eta/2)E)$,
$\varepsilon : \mathfrak{J}(4, \boldsymbol{H})^C \oplus \mathfrak{J}(4, \boldsymbol{H})^C \to \mathfrak{J}(4, \boldsymbol{H})^C \oplus \mathfrak{J}(4, \boldsymbol{H})^C$, $\varepsilon(M+iN, M'+iN')$
$= (M+iM', N+iN')$, where $M, N, M', N' \in \mathfrak{J}(4, \boldsymbol{H})$,
$h : \mathfrak{J}(4, \boldsymbol{H})^C \oplus \mathfrak{J}(4, \boldsymbol{H})^C \to \mathfrak{S}(8, C) \oplus \mathfrak{S}(8, C)$, $h(K, L) = (k(K)J, k(L)J)$.

Remark. put $(\mathfrak{P}^C)_\rho = \{P \in \mathfrak{P}^C \mid \rho P = P\}$. Then the complexification $((\mathfrak{P}^C)_\rho)^C = \{P_1 + iP_2 \mid P_1, P_2 \in (\mathfrak{P}^C)_\rho\}$ is $\mathfrak{P}^C$. Now, the restriction $\chi' = \chi \mid (\mathfrak{P}^C)_\rho : (\mathfrak{P}^C)_\rho \to \mathfrak{S}(8, C)$ of χ is

$$\chi'(P) = \chi'(X, Y, \xi, \eta) = k(gX-(\xi/2)E)J + ik(g(\gamma Y)-(\eta/2)E)J, \quad P \in (\mathfrak{P}^C)_\rho$$

and the original χ is the complexification of this χ'.

4. Homomorphism $\psi : SU(8) \to (E_7)_\rho$

We define a homomorphism $\psi : SU(8) \to (E_7)_\rho$ by

$$\psi(A)P = \chi^{-1}(A(\chi(P))^t A), \quad P \in \mathfrak{P}^C.$$

First of all, we must show $\psi(A) \in (E_7)_\rho$ for $A \in SU(8)$. To show this, it suffices to prove for their Lie algebras (because $SU(8)$ is connected), that is, the differential homomorphism $d\psi : \mathfrak{su}(8) \to \mathrm{Hom}_C(\mathfrak{P}^C, \mathfrak{P}^C)$ of ψ defined by

$$d\psi(D)P = \chi^{-1}(D(\chi(P)) + (\chi(P))^t D), \quad P \in \mathfrak{P}^C$$

coincides with the mapping $d\psi : \mathfrak{su}(8) \to (\mathfrak{e}_7)_\rho$ defined in Proposition 3.

(1) For $D = k(C)$, $C \in \mathfrak{sp}(4)$, $(X, Y, \xi, \eta) \in \mathfrak{P}^C$, $d\psi(k(C))(X, Y, \xi, \eta)$ is

$$\begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix} \xrightarrow{\gamma_2} \begin{bmatrix} X \\ \gamma Y \\ \xi \\ \eta \end{bmatrix} \xrightarrow{\tilde{g}} \begin{bmatrix} gX-(\xi/2)E \\ g(\gamma Y)-(\eta/2)E \end{bmatrix} \overset{\text{put}}{=} \begin{bmatrix} M+iN \\ M'+iN' \end{bmatrix}, \quad M, N, M', N' \in \mathfrak{J}(4, \boldsymbol{H}),$$

$$\xrightarrow{\varepsilon} \begin{bmatrix} M+iM' \\ N+iN' \end{bmatrix} \xrightarrow{h} \begin{bmatrix} k(M+iM')J \\ k(N+iN')J \end{bmatrix} \to \begin{bmatrix} k(C)k(M+iM')J + k(M+iM')J^t k(C) \\ k(C)k(N+iN')J + k(N+iN')J^t k(C) \end{bmatrix}$$

$$= \begin{bmatrix} k(C(M+iM'))J + k((M+iM')C^*)J \\ k(C(N+iN'))J + k((N+iN')C^*)J \end{bmatrix} \xrightarrow{h^{-1}} \begin{bmatrix} C(M+iM')+(M+iM')C^* \\ C(N+iN')+(N+iN')C^* \end{bmatrix}$$

$$\xrightarrow{\varepsilon^{-1}}\begin{bmatrix} C(M+iN)-(M+iN)C \\ C(M'+iN')-(M'+iN')C\end{bmatrix}=\begin{bmatrix} C(gX-(\xi/2)E)-(gX-(\xi/2)E)C) \\ C(g(\gamma Y)-(\eta/2)E)-(g(\gamma Y)-(\eta/2)E)C\end{bmatrix}$$

$$=\begin{bmatrix} C(gX)-(gX)C \\ C(g(\gamma Y))-(g(\gamma Y))C\end{bmatrix}\xrightarrow{\tilde{g}^{-1}}\begin{bmatrix} g^{-1}(C(gX)-(gX)C) \\ g^{-1}(C(g(\gamma Y))-(g(\gamma Y))C) \\ 0 \\ 0\end{bmatrix}\xrightarrow{\gamma_2^{-1}}\begin{bmatrix} d\varphi(C)X \\ \gamma d\varphi(C)\gamma Y \\ 0 \\ 0\end{bmatrix}$$

$$=\begin{bmatrix} d\varphi(C)X \\ \tau d\varphi(C)\tau Y \\ 0 \\ 0\end{bmatrix}=\begin{bmatrix} d\varphi(C) & 0 & 0 & 0 \\ 0 & \tau d\varphi(C)\tau & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} X \\ Y \\ \xi \\ \eta\end{bmatrix}=\Phi_\rho(d\varphi(C),0)\begin{bmatrix} X \\ Y \\ \xi \\ \eta\end{bmatrix}.$$

Thus we have $d\psi(k(C)=\Phi_\rho(d\varphi(C),0)$.

(2) For $D=ik(B)$, $B=gA=\begin{bmatrix} 0 & -p & 0 & 0 \\ -\bar{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}$, $A=iF_1(pe)$, $p\in \boldsymbol{H}$,

$d\psi(ik(B))(0,0,1,0)$ is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0\end{bmatrix}\xrightarrow{\tilde{g}\gamma_2}\begin{bmatrix} -E/2 \\ 0\end{bmatrix}\xrightarrow{\varepsilon}\begin{bmatrix} -E/2 \\ 0\end{bmatrix}\xrightarrow{h}\begin{bmatrix} -J/2 \\ 0\end{bmatrix}\rightarrow\begin{bmatrix} -(ik(B)J+J{}^ti{}^tk(B))/2 \\ 0\end{bmatrix}$$

$$=\begin{bmatrix} -ik(B)J \\ 0\end{bmatrix}\xrightarrow{h^{-1}}\begin{bmatrix} -iB \\ 0\end{bmatrix}\xrightarrow{\varepsilon^{-1}}\begin{bmatrix} 0 \\ -B\end{bmatrix}\xrightarrow{\tilde{g}^{-1}}\begin{bmatrix} 0 \\ -A \\ 0 \\ 0\end{bmatrix}\longrightarrow\begin{bmatrix} 0 \\ A \\ 0 \\ 0\end{bmatrix}$$

$$=\begin{bmatrix} 0 & 2A & 0 & A \\ 2A & 0 & A & 0 \\ 0 & A & 0 & 0 \\ A & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0\end{bmatrix}=\Phi_\rho(0,A)\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0\end{bmatrix}.$$

And $d\psi(ik(B))(X,0,0,0)$ is

$$\begin{bmatrix} X \\ 0 \\ 0 \\ 0\end{bmatrix}\overset{\text{put}}{=\!=}\begin{bmatrix}\begin{bmatrix} \xi_1 & a_3+b_3e & \bar{a}_2-b_2e \\ \bar{a}_3-b_3e & \xi_2 & a_1+b_1e \\ a_2+b_2e & \bar{a}_1-b_1e & \xi_3\end{bmatrix}+i\begin{bmatrix} \eta_1 & c_3+d_3e & \bar{c}_2-d_2e \\ \bar{c}_3-d_3e & \eta_2 & c_1+d_1e \\ c_2+d_2e & \bar{c}_2-d_1e & \eta_3\end{bmatrix} \\ 0 \\ 0 \\ 0\end{bmatrix}$$

$$\xrightarrow{\tilde{g}} \begin{bmatrix} \begin{bmatrix} \lambda_1 & -d_1 & -d_2 & -d_3 \\ -\bar{d}_1 & \lambda_2 & a_3 & \bar{a}_2 \\ -\bar{d}_2 & \bar{a}_3 & \lambda_3 & a_1 \\ -\bar{d}_3 & a_2 & \bar{a}_1 & \lambda_4 \end{bmatrix} + i \begin{bmatrix} \mu_1 & b_1 & b_2 & b_3 \\ \bar{b}_1 & \mu_2 & c_3 & \bar{c}_2 \\ \bar{b}_2 & \bar{c}_3 & \mu_3 & c_1 \\ \bar{b}_3 & c_2 & \bar{c}_1 & \mu_4 \end{bmatrix} \\ 0 \end{bmatrix} \overset{\text{put}}{=} \begin{bmatrix} M+iN \\ 0 \end{bmatrix}$$

$$\xrightarrow{\varepsilon} \begin{bmatrix} M \\ N \end{bmatrix} \xrightarrow{h} \begin{bmatrix} k(M)J \\ k(N)J \end{bmatrix} \rightarrow \begin{bmatrix} ik(B)k(M)J+k(M)J i{}^t k(B) \\ ik(B)k(N)J+k(N)J i{}^t k(B) \end{bmatrix} = \begin{bmatrix} ik(BM+MB)J \\ ik(BN+NB)J \end{bmatrix}$$

$$\xrightarrow{h^{-1}} \begin{bmatrix} i(BM+MB) \\ i(BN+NB) \end{bmatrix} \xrightarrow{\varepsilon^{-1}} \begin{bmatrix} 0 \\ (BM+MB)+i(BN+NB) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \begin{bmatrix} 2(p,\ d_1) & -\xi_1 p & -pa_3 & -p\bar{a}_2 \\ -\xi_1 \bar{p} & 2(p,\ d_1) & \bar{p}d_2 & \bar{p}d_3 \\ -\bar{a}_3\bar{p} & \bar{d}_2 p & 0 & 0 \\ -a_2\bar{p} & \bar{d}_3 p & 0 & 0 \end{bmatrix} + i \begin{bmatrix} -2(p,\ b_1) & -\eta_1 p & -pc_3 & -p\bar{c}_2 \\ -\eta_1\bar{p} & -2(p,\ b_1) & -\bar{p}b_2 & -\bar{p}b_3 \\ -\bar{c}_3\bar{p} & -\bar{b}_2 p & 0 & 0 \\ -c_2\bar{p} & -\bar{b}_3 p & 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$\xrightarrow{\tilde{g}^{-1}} \begin{bmatrix} 0 \\ \begin{bmatrix} 2(p,\ d_1) & \bar{p}d_2-(p\bar{c}_2)e & \bar{p}d_3+(pc_3)e \\ \bar{d}_2 p+(p\bar{c}_2)e & 0 & -\eta_1 pe \\ \bar{d}_3 p-(pc_3)e & \eta_1 pe & 0 \end{bmatrix} + i \begin{bmatrix} -2(p,\ b_1) & -\bar{p}b_2+(p\bar{a}_2)e & -\bar{p}b_3-(pa_2)e \\ -\bar{b}_2 p-(p\bar{a}_2)e & 0 & \xi_1 pe \\ -\bar{b}_3 p+(pa_3)e & -\xi_1 pe & 0 \end{bmatrix} \\ 0 \\ -2(p,\ d_1)+2i(p,\ b_1) \end{bmatrix}$$

$$\xrightarrow{\gamma_2^{-1}} \begin{bmatrix} 0 \\ \begin{bmatrix} 2(p,\ d_1) & \bar{p}d_2+(p\bar{c}_2)e & pd_3-(pc_3)e \\ \bar{d}_2 p-(p\bar{c}_2)e & 0 & \eta_1 pe \\ \bar{d}_3 p+(pc_3)e & -\eta_1 pe & 0 \end{bmatrix} + i \begin{bmatrix} -2(p,\ b_1) & -\bar{p}b_2-(p\bar{a}_2)e & -\bar{p}b_3+(pa_3)e \\ -b_2 p+(\bar{p}a_2)e & 0 & -\xi_1 pe \\ -\bar{b}_3 p-(pa_3)e & \xi_1 pe & 0 \end{bmatrix} \\ 0 \\ -2(p,\ d_1)+2i(p,\ b_1) \end{bmatrix}$$

$$\overset{**}{=} \begin{bmatrix} 0 \\ 2A\times X \\ 0 \\ (A,\ X) \end{bmatrix} = \begin{bmatrix} 0 & 2A & 0 & A \\ 2A & 0 & A & 0 \\ 0 & A & 0 & 0 \\ A & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ 0 \\ 0 \\ 0 \end{bmatrix} = \Phi_\rho(\phi,\ A) \begin{bmatrix} X \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (**,\ \text{in fact,}$$

$$\begin{aligned} 2A\times X &= 2iF_1(pe)\times(\textstyle\sum_{i=1}^3(\xi_i E_i+F_i(a_i+b_i e))+i\sum_{i=1}^3(\eta_i E_i+F_i(c_i+d_i e))) \\ &= i(-\xi_1 F_1(pe)-2(p,\ b_1)E_1+F_3(\overline{(pe)(a_2+b_2 e)})+F_2(\overline{(a_3+b_3 e)(pe)})) \\ &\quad -(-\eta_1 F_1(pe)-2(p,\ d_1)E_1+F_3(\overline{(pe)(c_2+d_2 e)})+F_2(\overline{(c_3+d_3 e)(pe)})) \\ &= (2(p,\ d_1)E_1+\eta_1 F_1(pe)+F_2(\bar{d}_3 p+(pc_3)e)+F_3(\bar{p}d_2+(p\bar{c}_2)e)) \\ &\quad +i(-2(p,\ b_1)E_1-\xi_1 F_1(pe)-F_2(\bar{b}_3 p+(pa_3)e)-F_3(\bar{p}b_2+(p\bar{a}_2 e)), \\ (A,\ X) &= (2iF_1(pe),\ (\textstyle\sum_{i=1}^3(\xi_i E_i+F_i(a_i+b_i e))+i\sum_{i=1}^3(\eta_i E_i+F_i(c_i+d_i e)))) \\ &= -2(p,\ d_1)+2i(p,\ b_1)). \end{aligned}$$

Similarly, we have $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \to \Phi_\rho(0, A) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ Y \\ 0 \\ 0 \end{bmatrix} \to \Phi_\rho(0, A) \begin{bmatrix} 0 \\ Y \\ 0 \\ 0 \end{bmatrix}$. Thus we have $d\psi(ik(B)) = \Phi_\rho(0, A) = \Phi_\rho(0, g^{-1}B)$.

(3) For other generators of $\mathfrak{su}(8)$, that is, for $D = ik(B)$, $B = gA$, where $A = iF_j(pe)$, $p \in \boldsymbol{H}$, $j = 2, 3$, $A = E_1 - E_j$, $j = 2, 3$ and $A = F_j(p)$, $P \in \boldsymbol{H}$, $j = 1, 2, 3$, we have also

$$d\psi(ik(B)) = \Phi_\rho(0, g^{-1}B)$$

by the straightforward calculations as the above (1), (2).

All together (1),(2),(3), we see that the homomorphism $\psi : SU(8) \to (E_7)_\rho$ is well-defined.

5. Isomorphism $(E_7)_\rho \cong SU(8)/Z_2$

Our aim of this section is to prove that $\psi : SU(8) \to (E_7)_\rho$ is onto.

Lemma 4. *For $a \in \boldsymbol{R}$, the linear transformation of $\mathfrak{P}^C$ defined by*

$$\alpha_i(a) = \begin{bmatrix} 1+(\cos|a|-1)p_i & (2a/|a|)\sin|a|E_i & 0 & -(a/|a|)\sin|a|E_i \\ -(2a/|a|)\sin|a|E_i & 1+(\cos|a|-1)p_i & (a/|a|)\sin|a|E_i & 0 \\ 0 & -(a/|a|)\sin|a|E_i & \cos|a| & 0 \\ (a/|a|)\sin|a|E_i & 0 & 0 & \cos|a| \end{bmatrix}$$

(if $a = 0$, then $(a/|a|)\sin|a|$ means 0) belongs to the group $\psi(SU(8))$, $i = 1, 2, 3$, where the mapping $p_i : \mathfrak{J}^C \to \mathfrak{J}^C$ is defined by

$$p_i \begin{bmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{bmatrix} = \begin{bmatrix} \xi_1 & \delta_{i3}x_3 & \delta_{i2}\bar{x}_2 \\ \delta_{i3}\bar{x}_3 & \xi_2 & \delta_{i1}x_1 \\ \delta_{i2}x_2 & \delta_{i1}\bar{x}_1 & \xi_3 \end{bmatrix}$$

(where δ_{ij} is the Kronecker's delta) and the action of $\alpha_i(a)$ on $\mathfrak{P}^C$ is defined as similar to that of $\Phi(\phi, A, B, \nu)$ in §1.3.

Proof. For $\Phi_\rho(0, -aE_i) \in d\psi(\mathfrak{su}(8))$, we have $\alpha_i(a) = \exp \Phi_\rho(0, -aE_i)$, hence $\alpha_i(a) \in \psi(SU(8))$, $i = 1, 2, 3$.

We define a subspace $(\mathfrak{M}^C)_\rho$ of $\mathfrak{P}^C$ by

$$(\mathfrak{M}^C)_\rho = \{P \in \mathfrak{M}^C \mid \rho P = P\}.$$

Lemma 5. *Any element $P \in (\mathfrak{M}^C)_\rho$ can be transformed in a real diagonal form by a certain element $\alpha \in \psi(SU(8))$:*

$\alpha P=(X, Y, \xi, \eta)$, *X, Y are real diagonal forms and* $\xi, \eta \in \boldsymbol{R}$. *Moreover we can choose* $\alpha \in \psi(SU(8))$ *so that* $\xi > 0$.

Proof. Let $P=(X, Y, \xi, \eta) \in (\mathfrak{M}^C)_\rho$. First assume that $\xi \neq 0$. Then $\rho Y = Y$, $\bar{\xi}=\xi$, $\bar{\eta}=\eta$ and $X=(Y\times Y)/\xi$. Since $g(\gamma Y)\in \mathfrak{J}(4, \boldsymbol{H})_0$ (Lemma 1 (2)), we can choose $C \in Sp(4)$ so that $C(g(\gamma Y))C^*$ is real diagonal, so $\gamma\phi(C)\gamma Y = g^{-1}(C(g(\gamma Y)C^*))$ has a real diagonal form. In this case, $\varphi(C)X=\varphi(C)((Y\times Y)/\xi)=(\gamma\varphi(C)\gamma Y\times \gamma\varphi(C)\gamma Y)/\xi$ is also real diagonal, hence $\psi(C)P$ is a diagonal form. In the case of $\eta \neq 0$, the statement is also valid. Next we consider the case $P=(X, Y, 0, 0)$, $Y\neq 0$. Choose $C\in Sp(4)$ such that $\gamma\varphi(C)\gamma Y=\begin{bmatrix}\eta_1 & 0 & 0\\ 0 & \eta_2 & 0\\ 0 & 0 & \eta_3\end{bmatrix}$, $\eta_i \in \boldsymbol{R}$. Since $\gamma\varphi(C)\gamma Y\neq 0$, we may assume $\eta_1 \neq 0$. Operate $\alpha_1(-\pi/2)\in\psi(SU(8))$ of Lemma 4 on $\psi(C)P$. Then

$$\alpha_1(-\pi/2)\psi(C)P=(*, *, \eta_1, *).$$

So, we can reduce to the first case $\xi\neq 0$. In the case of $P=(X, Y, 0, 0)$, $X\neq 0$, the statement is also valid. If $\xi<0$, then operate $\alpha_1(\pi)$ on αP. Then ξ becomes a positive number. Noting $\alpha_3(-\pi/2)\alpha_2(\pi/2)\alpha_1(\pi/2)(0, 0, 0, 1)=(0, 0, 1, 0)$, then we can always reduce to the case $\xi \neq 0$. Thus Lemma 5 is proved.

Now, we shall prove that $\psi: SU(8)\to (E_7)_\rho$ is onto. For a given $\alpha\in(E_7)_\rho$, consider an element $P=\alpha(0, 0, 1, 0)\in(\mathfrak{M})_\rho$. From Lemma 5, there exists $\beta\in\psi(SU(8))$ such that

$$\beta P=\left((1/\xi)\begin{bmatrix}\eta_2\eta_3 & 0 & 0\\ 0 & \eta_3\eta_1 & 0\\ 0 & 0 & \eta_1\eta_2\end{bmatrix}, \begin{bmatrix}\eta_1 & 0 & 0\\ 0 & \eta_2 & 0\\ 0 & 0 & \eta_3\end{bmatrix}, \xi, (\eta_1\eta_2\eta_3)/\xi^2\right),\ \xi>0,\ \eta_i\in\boldsymbol{R}.$$

Then the condition $\langle P, P\rangle = 1$ is

$$(1+(|\eta_1|/\xi)^2)(1+(|\eta_2|/\xi)^2)(1+(|\eta_3|/\xi)^2)=1/\xi^2.$$

Choose $r_i\in\boldsymbol{R}$, $\pi/2>r_i\geqq 0$, such that $\tan r_i=|\eta_i|/\xi$, $i=1, 2, 3$. Then we have

$$\xi=\cos r_1\cos r_2\cos r_3.$$

Put $a_i=(\eta_i/|\eta_i|)r_i$ (if $\eta_i=0$, then $(\eta_i/|\eta_i|)r_i$ means 0), $i=1, 2, 3$. Then we have

$$\beta P=\alpha_3(a_3)\alpha_2(a_2)\alpha_1(a_1)(0, 0, 1, 0)$$

(cf. [3, Theorem 9]), that is,

$$\alpha_1(a_1)^{-1}\alpha_2(a_2)^{-1}\alpha_3(a_3)^{-1}\beta\alpha(0,\ 0,\ 1,\ 0) = (0,\ 0,\ 1,\ 0).$$

Hence $\tilde{\alpha} = \alpha_1(a_1)^{-1}\alpha_2(a_2)^{-1}\alpha_3(a_3)^{-1}\beta\alpha \in E_6$, moreover $\rho\tilde{\alpha} = \tilde{\alpha}\rho$, therefore

$$\tilde{\alpha} = \alpha_1(a_1)^{-1}\alpha_2(a_2)^{-1}\alpha_3(a_3)^{-1}\beta\alpha \in (E_6)_\rho = \varphi(Sp(4)) \subset \psi(SU(8)).$$

Since $\alpha_i(a_i)$ and $\beta \in \psi(SU(8))$, α is also $\alpha \in \psi(SU(8))$, that is, ψ is onto.

Finally, $\operatorname{Ker}\psi = \{E,\ -E\}$ is easily obtained. Thus we have the following theorem which was our first aim.

Theorem 6. *The subgroup* $(E_7)_\rho = \{\alpha \in E_7 \mid \rho\alpha = \alpha\rho\}$ *of the group* E_7 *is isomorphic to the group* $SU(8)/\boldsymbol{Z}_2$, *where* $\boldsymbol{Z}_2 = \{E,\ -E\}$.

II. Lie group $E_{7(7)}$

6. Preliminaries

6.1. Split Cayley algebra $\mathfrak{C}'$ and split exceptional Jordan algebra $\mathfrak{J}'$. Let $\mathfrak{C}' = \boldsymbol{H} \oplus \boldsymbol{H}e$ denote the split Cayley algebra with the multiplication

$$(a+be)(c+de) = (ac+\bar{d}b)+(b\bar{c}+da)e.$$

And let $\mathfrak{J}' = \{X \in M(3,\ \mathfrak{C}') \mid X^* = X\}$ denote the split exceptional Jordan algebra with the multiplication $X \circ Y = (XY+YX)/2$. In $\mathfrak{J}'$ also, the inner product $(X,\ Y)' = \operatorname{tr}(X \circ Y)$, the crossed product $X \times Y$, the trilinear form $(X,\ Y,\ Z)' = (X,\ Y \times Z)'$, and the determinant $\det X = (X,\ X,\ X)'/3$ are defined as same as in $\mathfrak{J}$ and $\mathfrak{J}^C$. Moreover we define a positive definite inner product $(X,\ Y)$ in $\mathfrak{J}'$ by

$$(X,\ Y) = (X,\ \gamma Y)' = (\gamma X,\ Y)'$$

where $\gamma : \mathfrak{J}' \to \mathfrak{J}'$ is the involution defined as $\gamma : \mathfrak{J} \to \mathfrak{J}$ in §1.1.

6.2. Lie group $\boldsymbol{E_{6(6)}}$ and subgroup $\boldsymbol{(E_{6(6)})_K}$. We have shown in [6] that the group

$$E_{6(6)} = \{\alpha \in \operatorname{Iso}_{\boldsymbol{R}}(\mathfrak{J}',\ \mathfrak{J}') \mid \det\alpha X = \det X\}$$

is a connected non-compact simple Lie group of type $E_{6(6)}$ and found a subgroup of type C_4 in $E_{6(6)}$. To find this subgroup, we used a mapping $f : \mathfrak{J}' \to \mathfrak{J}(4,\ \boldsymbol{H})_0$,

$$f\begin{bmatrix} \xi_1 & a_3+b_3e & \bar{a}_2-b_2e \\ \bar{a}_3-b_3e & \xi_2 & a_1+b_1e \\ a_2+b_2e & \bar{a}_1-b_1e & \xi_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & b_1 & b_2 & b_3 \\ \bar{b}_1 & \lambda_2 & a_3 & \bar{a}_2 \\ \bar{b}_2 & \bar{a}_3 & \lambda_3 & a_1 \\ \bar{b}_3 & a_2 & \bar{a}_1 & \lambda_4 \end{bmatrix}$$

where $\xi_i \in \boldsymbol{R}$, a_i, $b_i \in \boldsymbol{H}$ and $\lambda_1 = (\xi_1+\xi_2+\xi_3)/2$, $\lambda_2 = (\xi_1-\xi_2-\xi_3)/2$, $\lambda_3 = (\xi_2-\xi_1-\xi_3)/2$, $\lambda_4 = (\xi_3-\xi_1-\xi_2)/2$.

Lemma 7 ([6, Lemma 1]). *The mapping* $f : \mathfrak{J}' \to \mathfrak{J}(4, \boldsymbol{H})_0$ *is a* $\boldsymbol{R}$*-isomorphism satisfying*

$$f(X \times Y) = f(\gamma X) \circ f(\gamma Y) - ((X, Y)/4)E.$$

Now, we have shown that a subgroup $(E_{6(6)})_K$ of $E_{6(6)}$

$$(E_{6(6)})_K = E_{6(6)} \cap O(\mathfrak{J}') = \{\alpha \in E_{6(6)} \mid (\alpha X, \alpha Y) = (X, Y)\}$$

is isomorphic to the group $Sp(4)/\boldsymbol{Z}_2$ by the correspondence

$$\varphi : Sp(4) \longrightarrow (E_{6(6)})_{K'} \quad \varphi(C)X = f^{-1}(C(fX)C^*), \quad X \in \mathfrak{J}'$$

with $\mathrm{Ker}\varphi = \boldsymbol{Z}_2 = \{E, -E\}$. Therefore its Lie algebra

$$(\mathfrak{e}_{6(6)})_K = \{\phi \in \mathfrak{e}_{6(6)} \mid (\phi X, Y) = -(X, \phi Y)\}.$$

is isomorphic to $\mathfrak{sp}(4)$ by the correspondence

$$d\varphi : \mathfrak{sp}(4) \longrightarrow (\mathfrak{e}_{6(6)})_{K'} \quad d\varphi(C)X = f^{-1}(C(fX)-(fX)C), \quad X \in \mathfrak{J}'.$$

Finally, we note that, for $A, B \in \mathfrak{J}'$, $A \vee B \in \mathfrak{e}_{6(6)}$ is defined by $(A \vee B)X = ((B, X)')/2)A + ((A, B)')/6)X - 2B \times (A \times X)$, $X \in \mathfrak{J}'$ analogously as $A \vee B \in \mathfrak{e}_6{}^C$ in § 1.2.

7. Lie group $E_{7(7)}$

Let $\mathfrak{P}'$ be a 56 dimensional vector space over $\boldsymbol{R}$ defined by

$$\mathfrak{P}' = \mathfrak{J}' \oplus \mathfrak{J}' \oplus \boldsymbol{R} \oplus \boldsymbol{R}.$$

In $\mathfrak{P}'$, the symmetric inner product $(P, Q)'$, the skew-symmetric inner product $\{P, Q\}'$ and one more positive definite inner product (P, Q) are defined respectively by

$$\begin{aligned}(P, Q)' &= (X, Z)' + (Y, W)' + \xi\zeta + \eta\omega,\\ \{P, Q\}' &= (X, W)' - (Y, Z)' + \xi\omega - \eta\zeta,\\ (P, Q) &= (X, Z) + (Y, W) + \xi\zeta + \eta\omega\end{aligned}$$

for $P = (X, Y, \xi, \eta)$, $Q = (Z, W, \zeta, \omega) \in \mathfrak{P}'$.

For $\phi \in \mathfrak{e}_{6(6)}$, $A, B \in \mathfrak{J}'$ and $\nu \in \boldsymbol{R}$, a linear transformation $\Phi(\phi, A, B, \nu)$ of $\mathfrak{P}'$ is defined by

$$\Phi(\phi, A, B, \nu)\begin{bmatrix} X \\ Y \\ \xi \\ \eta \end{bmatrix} = \begin{bmatrix} \phi X - (\nu/3)X + 2B \times Y + \eta A \\ 2A \times X + \phi' Y + (\nu/3)Y + \xi B \\ (A, Y)' + \nu\xi \\ (B, X)' - \nu\eta \end{bmatrix}$$

where $\phi' \in \mathfrak{e}_{6(6)}$ denotes the skew-transpose of ϕ with respect to the inner product $(X, Y)'$: $(\phi X, Y)' + (X, \phi' Y)' = 0$. For $P, Q \in \mathfrak{P}'$, a linear transformation

$$P \times Q = \Phi(\phi, A, B, \nu)$$

of $\mathfrak{P}'$ is analogously defined as $P \times Q$ in $\mathfrak{P}$ or $\mathfrak{P}^C$ (use $(X, Y)'$ instead of (X, Y)) and define a submanifold $\mathfrak{M}'$ of $\mathfrak{P}'$ by

$$\mathfrak{M}' = \{P \in \mathfrak{P}' \mid P \times P = 0, P \neq 0\}.$$

Now, we define a group $E_{7(7)}$ by

$$E_{7(7)} = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P}', \mathfrak{P}') \mid \alpha\mathfrak{M}' = \mathfrak{M}', \{\alpha P, \alpha Q\}' = \{P, Q\}'\}.$$

(Later, we see that this group $E_{7(7)}$ is connected (Theorem 13), therefore it may be also defined by (see [4])

$$E_{7(7)} = \{\alpha \in \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P}', \mathfrak{P}') \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q\}.)$$

The Lie algebra $\mathfrak{e}_{7(7)}$ of the group $E_{7(7)}$ is

$$\mathfrak{e}_{7(7)} = \{\Phi(\phi, A, B, \nu) \in \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{P}', \mathfrak{P}') \mid \phi \in \mathfrak{e}_{6(6)}, A, B \in \mathfrak{J}', \nu \in \boldsymbol{R}\}$$

(see [2],[3]). Since the complexification of the Lie algebra $\mathfrak{e}_{7(7)}$ is $\mathfrak{e}_7{}^C$, the group $E_{7(7)}$ is a simple Lie group of type E_7.

The group $E_{7(7)}$ contains a subgroup

$$\tilde{E}_{6(6)} = \left\{\alpha \in E_{7(7)} \,\middle|\, \begin{array}{l} \alpha(0, 0, 1, 0) = (0, 0, 1, 0) \\ \alpha(0, 0, 0, 1) = (0, 0, 0, 1) \end{array}\right\}$$

which is isomorphic to the group $E_{6(6)}$ by the correspondence

$$E_{6(6)} \ni \alpha \longleftrightarrow \alpha = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & {}^t\alpha^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \tilde{E}_{6(6)} \subset E_{7(7)}$$

where ${}^t\alpha$ is the transpose of $\alpha \in E_{6(6)}$ with respect to the inner product $(X, Y)'$: $(\alpha X, Y)' = (X, {}^t\alpha Y)'$.

In order to investigate the group $E_{7(7)}$, we consider a subgroup $(E_{7(7)})_K$ of $E_{7(7)}$:

$$(E_{7(7)})_K = E_{7(7)} \cap O(\mathfrak{P}') = \{\alpha \in E_{7(7)} \mid (\alpha P, \alpha Q) = (P, Q)\}$$

Proposition 8. *The Lie algebra* $(\mathfrak{e}_{7(7)})_K$ *of the group* $(E_{7(7)})_K$ *is*

$$\begin{aligned}(\mathfrak{e}_{7(7)})_K &= \{\Phi \in \mathfrak{e}_{7(7)} \mid (\Phi P, Q) = -(P, \Phi Q)\} \\ &= \{\Phi'(\phi, A) \in \mathfrak{e}_{7(7)} \mid \phi \in (\mathfrak{e}_{6(6)})_K, A \in \mathfrak{J}'\} \\ &= \{\Phi'(d\varphi(C), f^{-1}B) \in \mathfrak{e}_{7(7)} \mid C \in \mathfrak{sp}(4), B \in \mathfrak{J}(4, \boldsymbol{H})_0\}\end{aligned}$$

where $\Phi'(\phi, A) = \Phi(\phi, A, -\gamma A, 0)$. *And it is isomorphic to the special unitary Lie algebra* $\mathfrak{su}(8)$ *by the correspondence*

$$k(C)+ik(B) \in \mathfrak{su}(8) \xrightarrow{d\psi} \Phi'(d\varphi(C), f^{-1}B) \in (\mathfrak{e}_{7(7)})_K$$

where $C \in \mathfrak{sp}(4)$, $B \in \mathfrak{J}(4, \boldsymbol{H})_0$.

Proof. The proof is similar to that of Proposition 3 by using Lemma 7.

8. $\boldsymbol{R}$-isomorphism χ between $\mathfrak{P}'$ and $\mathfrak{S}(8, \boldsymbol{C})$

We give an $\boldsymbol{R}$-isomorphism $\chi : \mathfrak{P}' \to \mathfrak{S}(8, \boldsymbol{C})$ by

$$\chi = h\varepsilon\tilde{f}\gamma_2$$

(cf. Remark of §3), where γ_2, $\tilde{f}$, ε, h are $\boldsymbol{R}$-isomorphisms defined by

$\gamma_2 : \mathfrak{P}' \to \mathfrak{P}'$, $\gamma_2(X, Y, \xi, \eta) = (X, \gamma Y, \xi, \eta)$,
$\tilde{f} : \mathfrak{P}' \to \mathfrak{J}(4, \boldsymbol{H}) \oplus \mathfrak{J}(4, \boldsymbol{H})$, $\tilde{f}(X, Y, \xi, \eta) = (fX-(\xi/2)E, fY-(\eta/2)E)$,
$\varepsilon : \mathfrak{J}(4, \boldsymbol{H}) \oplus \mathfrak{J}(4, \boldsymbol{H}) \to \mathfrak{J}(4, \boldsymbol{H})^C$, $\varepsilon(M, N) = M+iN$,
$h : \mathfrak{J}(4, \boldsymbol{H})^C \to \mathfrak{S}(8, \boldsymbol{C})$, $h(L) = k(L)J$.

9. Homomorphism $\psi : SU(8) \to (E_{7(7)})_K$

We define a homomorphism $\psi : SU(8) \to (E_{7(7)})_K$ by

$$\psi(A)P = \chi^{-1}(A(\chi(P))^tA) \qquad P \in \mathfrak{P}'.$$

To show that $\psi(A) \in (E_{7(7)})_K$ for $A \in SU(8)$, it suffices to prove for their Lie algebras, that is, the differential homomorphism $d\psi : \mathfrak{su}(8) \to \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{P}', \mathfrak{P}')$ of ψ:

$$d\psi(D)P = \chi^{-1}(D(\chi(P))+\chi(P)^tD) \qquad P \in \mathfrak{P}'$$

coincides with the mapping $d\psi : \mathfrak{su}(8) \to (\mathfrak{e}_{7(7)})_K$ defined in Proposition 8. However the proof is the same as the proof of the section 4.

10. Isomorphism $(E_{7(7)})_K \cong SU(8)/Z_2$

We shall show that $\psi : SU(8) \to (E_{7(7)})_K$ is onto. For this purpose, if we prepare the following three lemmas, we can attain our aim by using the same way as the section 5.

Lemma 9. *For* $a \in \boldsymbol{R}$, *linear transformations* $\alpha_i(a)$ *of* $\mathfrak{P}'$ *defined analogously as Lemma* 4 *belongs to* $\psi(SU(8))$, $i = 1, 2, 3$.

Lemma 10. *Any element* $P \in (\mathfrak{M}')_K = \{P \in \mathfrak{M}' \mid (P, P) = 1\}$ *can be transformed in a diagonal form by a certain element* $\alpha \in \psi(SU(8))$:

$aP = (X, Y, \xi, \eta)$ $\quad X, Y$ *are diagonal forms*, $\xi > 0$, $\eta \in \boldsymbol{R}$.

Lemma 11. *If* $\alpha \in (E_{7(7)})_K$ *satisfies* $\alpha(0, 0, 1, 0) = (0, 0, 1, 0)$, *then* $\alpha(0, 0, 0, 1) = (0, 0, 0, 1)$.

Now, since it is easy to see that the kernel of $\psi : SU(8) \to (E_{7(7)})_K$ is $\{E, -E\}$, thus we have

Proposition 12. *The group* $(E_{7(7)})_K = \{\alpha \in E_{7(7)} \mid (\alpha P, \alpha Q) = (P, Q)\}$ *is isomorphic to the group* $SU(8)/\boldsymbol{Z}_2$, *where* $\boldsymbol{Z}_2 = \{E, -E\}$.

11. Polar decomposition of $E_{7(7)}$

We define a linear transformation ι' of $\mathfrak{P}'$ by

$$\iota'(X, Y, \xi, \eta) = (Y, -X, \eta, -\xi).$$

Then $\iota' \in E_{7(7)}$, $\iota'^2 = -1$ and $\{P, Q\}' = -(\iota P, Q)' = (P, \iota Q)'$ for $P, Q \in \mathfrak{P}'$. Therefore put

$$\upsilon = \gamma\iota' = \iota'\gamma.$$

Then $\upsilon \in E_{7(7)}$, $\upsilon^2 = -1$ and we have

$$\{P, Q\}' = -(\upsilon P, Q) = (P, \upsilon Q), \qquad P, Q \in \mathfrak{P}'.$$

To give a polar decomposition of the group $E_{7(7)}$, we prepare

Lemma 13. *The group* $E_{7(7)}$ *is an algebraic subgroup of the general linear group* $GL(56, \boldsymbol{R}) = \mathrm{Iso}_{\boldsymbol{R}}(\mathfrak{P}', \mathfrak{P}')$ *and satisfies the condition that* $\alpha \in E_{7(7)}$ *implies* ${}^t\alpha \in E_{7(7)}$, *where* ${}^t\alpha$ *is the transpose of* α *with respect to the inner product* $(P, Q) : (\alpha P, Q) = (P, {}^t\alpha Q)$.

Proof. Since $({}^t\alpha P, Q) = (P, \alpha Q) = \{\upsilon P, \alpha Q\}' = \{\alpha^{-1}\upsilon P, Q\}' = -(\upsilon\alpha^{-1}\upsilon P, Q)$ for $\alpha \in E_{7(7)}$, we have

$${}^t\alpha = -\upsilon\alpha^{-1}\upsilon \in E_{7(7)}.$$

And it is obvious that the group $E_{7(7)}$ is algebraic, because it is defined by algebraic relations $\alpha\mathfrak{M}' = \mathfrak{M}'$, $\{\alpha P, \alpha Q\}' = \{P, Q\}'$.

Using Chevalley's lemma ([1, Lemma 2, p. 201]), we have a homeomorphism

$$\begin{aligned} E_{7(7)} &\simeq (E_{7(7)} \cap O(\mathfrak{P}')) \times \boldsymbol{R}^d \\ &\simeq SU(8)/\boldsymbol{Z}_2 \times \boldsymbol{R}^d \qquad \text{(Proposition 12).} \end{aligned}$$

Since $E_{7(7)}$ is a simple Lie group of type E_7, the dimension of $E_{7(7)}$ is 133. Hence the dimension d of the Euclidean part and the Cartan index i of $E_{7(7)}$ are calculated respectively as follows:

$$d = \dim E_{7(7)} - \dim SU(8) = 133-63 = 70,$$
$$i = \dim E_{7(7)} - 2\dim SU(8) = 133-2\times 63 = 7.$$

Thus we have the following theorem which was our second aim.

Theorem 14. *The group $E_{7(7)}$ is homeomorphic to the topological product of the group $SU(8)/\boldsymbol{Z}_2$ and a Euclidean space $\boldsymbol{R}^{70}$:*

$$E_{7(7)} \simeq SU(8)/\boldsymbol{Z}_2 \times \boldsymbol{R}^{70}.$$

In particular, $E_{7(7)}$ is a connected non-compact simple Lie group of type $E_{7(7)}$.

12. Center $z(E_{7(7)})$ of $E_{7(7)}$

Theorem 15. *The center $z(E_{7(7)})$ of the group $E_{7(7)}$ is the cyclic group of order 2:*

$$z(E_{7(7)}) = \{1, \ -1\}.$$

Proof. Let $\alpha \in z(E_{7(7)})$. From the commutativity with $\upsilon \in E_{7(7)}$, we have ${}^t\alpha = -\upsilon\alpha^{-1}\upsilon = -\upsilon\upsilon\alpha^{-1} = \alpha^{-1}$. Hence $\alpha \in (E_{7(7)})_{K'}$ therefore there exists $A \in SU(8)$ such that $\alpha = \psi(A)$. Furthermore

$\alpha \in z((E_{7(7)})_K) = z(\psi(SU(8)))$ ($z(G)$ denotes always the center of a group G and in this case we have)

$$= \psi(z(SU(8)))$$
$$= \psi\{p^iE, \ i = 0, 1, 2, \cdots, 7\}, \ p = e^{i\pi/4}$$
$$= \psi\{E, \ pE, \ p^2E, \ p^3E\}$$
$$= \{1, \ -\upsilon, \ -1, \ \upsilon\} \text{ (because } \psi(pE) = -\upsilon).$$

However, υ, $-\upsilon$ are not contained in $z(E_{7(7)})$. In fact, for $r \in \boldsymbol{R}, r \neq 0$, define a linear transformation $\boldsymbol{r}$ of $\mathfrak{P}'$ by

$$\boldsymbol{r}(X, \ Y, \ \xi, \ \eta) = (r^{-1}X, \ rY, \ r^3\xi, \ r^{-3}\eta).$$

Then $\boldsymbol{r} \in E_{7(7)}$ and $\boldsymbol{r}\upsilon \neq \upsilon\boldsymbol{r}$, $\boldsymbol{r}(-\upsilon) \neq (-\upsilon)\boldsymbol{r}$ for $r \neq 1$, because $\boldsymbol{r}\upsilon(0, 0, 1, 0) = \boldsymbol{r}(0, 0, 0, -1) = (0, 0, 0, -r^{-3})$, $\upsilon\boldsymbol{r}(0, 0, 1, 0) = \upsilon(0, 0, r^3, 0) = (0, 0, 0, -r^3)$. Thus we have $z(E_{7(7)}) = \{1, \ -1\}$.

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