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ON J-GROUPS OF $S^l(RP(t\text{-}l)/RP(n\text{-}l))$

SUSUMU KÔNO and AKIE TAMAMURA

1. Introduction. Let t, n and l be non-negative integers with $t > n \geqq l$. We write $S^l(RP(t-l)/RP(n-l))$ for the l-times suspension of the stunted real projective space $RP(t-l)/RP(n-l)$. The purpose of this paper is to determine the J-group of $S^l(RP(t-l)/RP(n-l))$. The partial result for $n = 2l$ is obtained in [3, Theorem 3.5].

In order to state our theorem, we recall some notation in [1]: $\phi(n_1, n_2)$ is the number of integers s with $n_2 < s \leqq n_1$ and $s \equiv 0, 1, 2$ or $4 \pmod 8$, $\nu_p(s)$ denotes the exponent of the prime p in the prime power decomposition of s and $\mathrm{m}(s)$ is the function defined on positive integers as follows:

$$\nu_p(\mathrm{m}(s)) = \begin{cases} 0 & \text{if } p \neq 2,\ s \not\equiv 0 \pmod{(p-1)} \\ 1+\nu_p(s) & \text{if } p \neq 2,\ s \equiv 0 \pmod{(p-1)} \\ 1 & \text{if } p = 2,\ s \not\equiv 0 \pmod 2 \\ 2+\nu_2(s) & \text{if } p = 2,\ s \equiv 0 \pmod 2. \end{cases}$$

Then our theorem is stated as follows.

Theorem. (1) *If* $l \equiv 0 \pmod 4$ *and* $n \not\equiv 3 \pmod 4$, *then*

$$\tilde{J}(S^l(RP(t-l)/RP(n-l))) \cong Z_{2^h},$$

where

$$h = \begin{cases} \min\{\phi(t, n), \nu_2(l)+1\} & (l > 0) \\ \phi(t, n) & (l = 0). \end{cases}$$

(2) *If* $l \equiv 0 \pmod 4$ *and* $n \equiv 3 \pmod 4$, *then*

$$\tilde{J}(S^l(RP(t-l)/RP(n-l))) \cong Z_{2^i} \oplus Z_K,$$

where

$$i = \begin{cases} \min\{\phi(t, n+1), \nu_2(l)\} & (l > 0,\ \nu_2(l) < \nu_2(n+1)) \\ \min\{\phi(t, n+1), \nu_2(l)+1\} & (l > 0,\ \nu_2(l) = \nu_2(n+1)) \\ \min\{\phi(t, n+1), \nu_2(n+1)\} & (\textit{otherwise}), \end{cases}$$

and

$$K = \begin{cases} \mathrm{m}\left(\dfrac{n+1}{2}\right) \cdot 2^{\min\{\phi(t,n+1), \nu_2(l)+1\}-i} & (l > 0) \\ \mathrm{m}\left(\dfrac{n+1}{2}\right) \cdot 2^{\phi(t,n+1)-i} & (l = 0). \end{cases}$$

(3) *If* $l \not\equiv 0$ (mod 4) *and* $t \geqq n+3$, *then the groups* $\tilde{J}(S^l(RP(t-l)/RP(n-l)))$ *are tabled as follows, where* $M = \mathrm{m}\left(\frac{t}{2}\right)$, $L = \mathrm{m}\left(\frac{n+1}{2}\right)$ *and* $(n_1, \cdots, n_j)$ *denotes the direct sum* $Z_{n_1} \oplus \cdots \oplus Z_{n_j}$.

(i) $l \equiv 1$ (mod 4)

	n (mod 8) \ t (mod 8)	0	1	2	3	4	5	6	7
1)	0	$(M,2,2)$	(2,2)	(2,2)	(2,2)	$(M,2,2)$	(2,2)	(2,2)	(2,2)
2)	1	$(M,2)$	(2)	(2)	(2)	$(M,2)$	(2)	(2)	(2)
3)	2	(M)	0	0	0	(M)	0	0	0
4)	3	(M)	0	0	0	(M)	0	0	0
5)	4	(M)	0	0	0	(M)	0	0	0
6)	5	(M)	0	0	0	(M)	0	0	0
7)	6	(M)	0	0	0	(M)	0	0	0
8)	7	$(M,2)$	(2)	(2)	(2)	$(M,2)$	(2)	(2)	(2)

(ii) $l \equiv 2$ (mod 4)

	n (mod 8) \ t (mod 8)	0	1	2	3	4	5	6	7
9)	0	(2,2)	(2,2,2)	(2,2)	(2)	(2)	(2)	(2)	(2)
10)	1	(2,2,2)	(2,2,2,2)	(2,2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)
11)	2	(2,2)	(2,2,2)	(2,2)	(2)	(2)	(2)	(2)	(2)
12)	3	$(L,2)$	$(L,2,2)$	$(L,2)$	(L)	(L)	(L)	(L)	(L)
13)	4	(2)	(2,2)	(2)	0	0	0	0	0
14)	5	(2)	(2,2)	(2)	0	0	0	0	0
15)	6	(2)	(2,2)	(2)	0	0	0	0	0
16)	7	$(L,2)$	$(L,2,2)$	$(L,2)$	(L)	(L)	(L)	(L)	(L)

(iii) $l \equiv 3 \pmod 4$

	t (mod 8) / n (mod 8)	0	1	2	3	4	5	6	7
17)	0	(M)	(2)	(2,2)	(2)	(M)	0	0	0
18)	1	(M)	(2)	(2,2)	(2)	(M)	0	0	0
19)	2	(M)	(2)	(2,2)	(2)	(M)	0	0	0
20)	3	(M)	(2)	(2,2)	(2)	(M)	0	0	0
21)	4	(M)	(2)	(2,2)	(2)	(M)	0	0	0
22)	5	(M)	(2)	(2,2)	(2)	(M)	0	0	0
23)	6	(M)	(2)	(2,2)	(2)	(M)	0	0	0
24)	7	(M)	(2)	(2,2)	(2)	(M)	0	0	0

Remark. (a) If $t = n+1$, then $S^l(RP(t-l)/RP(n-l)) \approx S^t$ and $\tilde{J}(S^t)$ is determined by Adams [1].

(b) If $t = n+2$, then we have homotopy equivalences

$$S^l(RP(t-l)/RP(n-l)) \simeq \begin{cases} S^t \vee S^{t-1} & (t-l : \text{odd}) \\ S^{t-2}(RP(2)) & (t-l : \text{even}), \end{cases}$$

and part (3) is true except the following cases:

i) If $l \equiv 2 \pmod 4$ and $t = n+2 \equiv 2 \pmod 8$, then
$$\tilde{J}(S^l(RP(t-l)/RP(n-l))) \cong Z_4.$$

ii) If $l \equiv 2 \pmod 4$ and $t = n+2 \equiv 3 \pmod 8$, then
$$\tilde{J}(S^l(RP(t-l)/RP(n-l))) \cong Z_2.$$

iii) If $l \equiv 2 \pmod 4$ and $t = n+2 \equiv 1 \pmod 8$, then
$$\tilde{J}(S^l(RP(t-l)/RP(n-l))) \cong Z_{m(\frac{n+1}{2})} \oplus Z_2.$$

The paper is organized as follows. In §2 we give a proof of (1) and (2). In §3 we give a proof of (3).

2. Proofs of (1) and (2). In this section we put $X = S^l(RP(t-l)/RP(n-l))$ for convenience' sake. We begin with part (1). The group $\widetilde{KO}(X)$ is a cyclic group of order $2^{\varphi(t,n)}$, and the Adams operations ψ^k on $\widetilde{KO}(X)$ are given by

$$\psi^k = \begin{cases} 0 & (k : \text{even}) \\ k^{l/2} & (k : \text{odd}). \end{cases} \tag{2.1}$$

Therefore we have $\sum_k (\bigcap_e k^e(\psi^k - 1)\widetilde{KO}(X) = \sum_{k:\text{odd}} (k^{t/2}-1)\widetilde{KO}(X)$.

According to [3, Lemma 3.3] the greatest common divisor of $2^{\varphi(t,n)}$ and the integers $k^{t/2}-1$ (k : odd) equals 2^h. Therefore

$$\tilde{J}(X) \cong \widetilde{KO}(X)/\sum_k (\bigcap_e k^e(\psi^k-1)\widetilde{KO}(X) \cong Z_{2^h}.$$

This proves part (1).

We now turn to part (2). Consider the commutative diagram

(2.2)

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \widetilde{KO}(S^l(RP(t-l)/RP(n+1-l))) & = & \widetilde{KO}(S^l(RP(t-l)/RP(n+1-l))) & & \\
 & & \downarrow j & & \downarrow & & \\
0 \longrightarrow \widetilde{KO}(S^{n+1}) & \xrightarrow{\delta_1} & \widetilde{KO}(X) & \xrightarrow{j_1} & \widetilde{KO}(S^l(RP(t-l)/RP(n-1-l))) & \longrightarrow & 0 \\
\| & & \downarrow i & & \downarrow & & \\
0 \longrightarrow \widetilde{KO}(S^{n+1}) & \xrightarrow{\delta_2} & \widetilde{KO}(S^l(RP(n+1-l)/RP(n-l))) & \longrightarrow & \widetilde{KO}(S^l(RP(n+1-l)/RP(n-1-l))) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

of exact sequences.

Recall from [2] that $\widetilde{KO}(S^{n+1}) \cong \widetilde{KO}(S^l(RP(n+1-l)/RP(n-l))) \cong Z$, $\widetilde{KO}(S^l(RP(t-l)/RP(n+1-l))) \cong Z_{2^{\varphi(t,n-1)}}$, $\widetilde{KO}(S^l(RP(t-l)/RP(n-1-l))) \cong Z_{2^{\varphi(t,n+1)-1}}$ and $\widetilde{KO}(S^l(RP(n+1-l)/RP(n-1-l))) \cong Z_2$.

Let α be a generator of $\widetilde{KO}(S^l(RP(t-l)/RP(n+1-l)))$, and set $a = j(\alpha)$. Let β be a generator of $\widetilde{KO}(S^l(RP(n+1-l)/RP(n-l)))$. Then $\widetilde{KO}(S^{n+1})$ has generator γ with $\delta_2(\gamma) = 2\beta$. If $b \in \widetilde{KO}(X)$ satisfies $i(b) = \beta$, an isomorpism

$$\theta : \widetilde{KO}(S^l(RP(t-l)/RP(n+1-l))) \oplus \widetilde{KO}(S^l(RP(n+1-l)/RP(n-l))) \longrightarrow \widetilde{KO}(X)$$

is defined by $\theta(p\alpha, q\beta) = pa + qb$ ($p, q \in Z$). (Note that generally θ is not a ψ-map in the sense of Adams [1].) Inspecting the diagram (2.2), we see that we can assume that $\delta_1(\gamma) = a + 2b$ without loss of generality. Choose $b \in \widetilde{KO}(X)$ such that $i(b) = \beta$ and $\delta_1(\gamma) = a + 2b$.

Lemma 2.3. *The Adams operations ψ^k on $\widetilde{KO}(X)$ is given by the following formulae*:

$$\psi^k(a) = \begin{cases} 0 & (k : \text{even}) \\ k^{t/2}a & (k : \text{odd}), \end{cases}$$

$$\psi^k(b) = k^{(n+1)/2}b + \begin{cases} \frac{1}{2}k^{(n+1)/2}a & (k : \text{even}) \\ \frac{1}{2}(k^{(n+1)/2} - k^{t/2})a & (k : \text{odd}). \end{cases}$$

Proof. Since j commutes with the Adams operation, the result about $\psi^k(a)$ follows from (2.1).

We necessarily have

$$\psi^k(b) = ua+vb$$

for some integers u, v. By using the ψ-map i, we see that $v = k^{(n+1)/2}$. Now project into $S^l(RP(t-l)/RP(n-1-l))$; a maps into $-2j_1(b)$, and we see that

$$\psi^k(j_1(b)) = (-2u+v)j_1(b).$$

Therefore

$$u \equiv \frac{1}{2}(k^{(n+1)/2}-\varepsilon k^{l/2}) \pmod{2^{\phi(t,n+1)}},$$

where $\varepsilon = 0$ or 1 according as k is even or odd. Thus we have the lemma. q.e.d.

We now recall some definition in [1]. Let f be a function which assigns to each integer k a non-negative integer $f(k)$. Given such a function f, we define $\widetilde{KO}(X)_f$ to be the subgroup of $\widetilde{KO}(X)$ generated by $\{k^{f(k)}(\psi^k-1)x \mid k \in Z,\ x \in \widetilde{KO}(X)\}$:

$$\widetilde{KO}(X)_f = \langle\{k^{f(k)}(\psi^k-1)x \mid k \in Z,\ x \in \widetilde{KO}(X)\}\rangle.$$

Then the kernel of the homomorphism $J: \widetilde{KO}(X) \longrightarrow \tilde{J}(X)$ coincides with $\bigcap_f \widetilde{KO}(X)_f$. By virtue of [1, (2.7) and (2.12)], Lemma 2.3 shows that $\bigcap_f \widetilde{KO}(X)_f = \langle\{M_2a,\ M_1(a+2b),\ \frac{1}{2}M_2a+\frac{1}{2}M_1(a+2b)\}\rangle = \langle\{M_2a,\ \frac{1}{2}(M_1+M_2)a+M_1b\}\rangle$, where $M_1 = \mathrm{m}\left(\frac{n+1}{2}\right)$ and M_2 is defined by

$$M_2 = \begin{cases} 0 & (l=0) \\ \mathrm{m}\left(\frac{l}{2}\right) & (l \neq 0). \end{cases}$$

Therefore $\tilde{J}(X) \cong \widetilde{KO}(X)/\langle\{M_2a,\ \frac{1}{2}(M_1+M_2)a+M_1b\}\rangle$. Since a has the order $2^{\phi(t,n+1)}$, an easy calculation shows that $J(a)$ (resp. $J(b)$) has the order L (resp. $LM_1/2^i$), where L is the greatest common divisor of M_2 and $2^{\phi(t,n+1)}$. Hence the order of $\tilde{J}(X)$ equals to LM_1 and the exponent of $\tilde{J}(X)$ equals to $LM_1/2^i$. Since $\tilde{J}(X)$ is generated by $J(a)$ and $J(b)$, this proves part (2).

3. Proof of (3). In the same way as [3, Theorem 3.5], we have the parts 3), 4), 5), 6), 7), 13), 14), 15), 17), 18), 19), 20), 21), 22), 23) and 24) of the table.

Let $l \equiv 2 \pmod 4$ and $n \equiv 3 \pmod 8$. Choose an integer $s \geqq t$ with $s \equiv 7 \pmod 8$. Since $\widetilde{KO}(S^{l+1}(RP(n-l+1)/RP(n-l))) = \widetilde{KO}(S^l(RP(s-l)/RP(n-l+1))) = \widetilde{KO}(S^{l-1}(RP(s-l)/RP(n-l+1))) = 0$, we have a commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow \widetilde{KO}(S^l(RP(t-l)/RP(n-l+1))) & \longrightarrow & \widetilde{KO}(S^l(RP(t-l)/RP(n-l))) & \longrightarrow & \widetilde{KO}(S^{n+1}) & & \\ & & \uparrow & & \| & & \\ 0 & \longrightarrow & \widetilde{KO}(S^l(RP(s-l)/RP(n-l))) & \longrightarrow & \widetilde{KO}(S^{n+1}) & \longrightarrow & 0 \end{array}$$

of ψ-groups with exact rows. This implies that the sequence

$$0 \longrightarrow \widetilde{KO}(S^l(RP(t-l)/RP(n-l+1))) \longrightarrow \widetilde{KO}(S^l(RP(t-l)/RP(n-l))) \longrightarrow \widetilde{KO}(S^{n+1}) \longrightarrow 0$$

of ψ-groups is exact and splits. Since $\tilde{J}(S^{n+1}) \cong Z_{m(\frac{n+1}{2})}$, it follows from [3, Lemma 2.2] that we have

$$\tilde{J}(S^l(RP(t-l)/RP(n-l))) \cong \tilde{J}(S^l(RP(t-l)/RP(n-l+1))) \oplus Z_{m(\frac{n+1}{2})}$$

Thus, 12) follows from 13). Similarly, 1) and 2) follow from 3), and 11) follows from 13).

Let $l \equiv 2 \pmod 4$ and $n \equiv 1 \pmod 8$. Since

$$\widetilde{KO}(S^{l+1}(RP(n-l+3)/RP(n-l-3))) = \widetilde{KO}(S^l(RP(n-l+3)/RP(n-l-3))) = 0,$$

we have a commutative diagram

$$\begin{array}{ccccc} 0 \longrightarrow \widetilde{KO}(S^{l+1}(RP(n-l)/RP(n-l-3))) & \longrightarrow & \widetilde{KO}(S^l(RP(n-l+3)/RP(n-l))) & & \\ \| & & \uparrow & & \\ \widetilde{KO}(S^{l+1}(RP(n-l)/RP(n-l-3))) & \longrightarrow & \widetilde{KO}(S^l(RP(t-l)/RP(n-l))) & & \\ \longrightarrow 0 & \longrightarrow & \widetilde{KO}(S^l(RP(n-l)/RP(n-l-3))) & & \\ \uparrow & & \| & & \\ \longrightarrow \widetilde{KO}(S^l(RP(t-l)/RP(n-l-3))) & \longrightarrow & \widetilde{KO}(S^l(RP(n-l)/RP(n-l-3))) & & \end{array}$$

of ψ-groups with exact rows. This implies that the sequence

$$0 \longrightarrow \widetilde{KO}(S^{l+1}(RP(n-l)/RP(n-l-3))) \longrightarrow \widetilde{KO}(S^l(RP(t-l)/RP(n-l))) \longrightarrow \widetilde{KO}(S^l(RP(t-l)/RP(n-l-3))) \longrightarrow 0$$

of ψ-groups is exact and splits. Since $\tilde{J}(S^{l+1}(RP(n-l)/RP(n-l-3))) \cong Z_2 \oplus Z_2$ by 24), we have

$$\tilde{J}(S^l(RP(t-l)/RP(n-l))) \cong \tilde{J}(S^l(RP(t-l)/RP(n-l-3))) \oplus Z_2 \oplus Z_2.$$

Thus 10) follows from 15). The rest is similar to the above. This completes the proof.

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OKAYAMA UNIVERSITY
OKAYAMA UNIVERSITY OF SCIENCE

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