Math. J. Okayama Univ. 24 (1982), 37—44

ON THE ITERATED SAMELSON PRODUCT

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1. Introduction. Let G be a topological group. The Samelson product $\langle\ ,\ \rangle$ of G is a pairing

$$\pi_p(G)\times\pi_q(G)\longrightarrow\pi_{p+q}(G)\qquad(p,q\geqq 1)$$

defined as follows. Let $\alpha\in\pi_p(G)$, $\beta\in\pi_q(G)$ be represented by maps

$$f:(I^p,\dot{I}^p)\longrightarrow(G,e),\qquad g:(I^q,\dot{I}^q)\longrightarrow(G,e).$$

Then $\langle\alpha,\beta\rangle\in\pi_{p+q}(G)$ is defined to be the element represented by the map

$$k:(I^p\times I^q,I^p\times\dot{I}^q\cup\dot{I}^p\times I^q)\longrightarrow(G,e),$$

where $k(x,y)=f(x)g(y)f(x)^{-1}g(y)^{-1}$ for $x\in I^p$ and $y\in I^q$.

With each element $\lambda\in\pi_1(G)$ we associate the operator

$$\boldsymbol{D}_\lambda:\pi_r(G)\longrightarrow\pi_{r+1}(G)$$

defined by taking the Samelson product with λ. From the Jacobi identity, each of these operators constitutes a derivation, with respect to the Samelson product in $\pi_*(G)$. Suppose that $\pi_2(G)=0$, as in the case when G is a Lie group. Then $2\boldsymbol{D}_\lambda^2=0$; moreover, there is some evidence to support the following

Conjecture (I. M. James [6]). For some value of s, depending on λ but not on r, the operator

$$\boldsymbol{D}_\lambda^s:\pi_r(G)\longrightarrow\pi_{r+s}(G),$$

defined by iteration of $\boldsymbol{D}_\lambda$, is trivial.

Note that $\boldsymbol{D}_\lambda=0$ if λ can be represented by a loop within the center of G or G is commutative. In [5], James proved that the conjecture is true for the rotation group R_t and the generator of $\pi_1(R_t)\cong\boldsymbol{Z}_2$.

In this paper we show the corresponding result for the unitary group and the generator of its fundamental group by making use of the relation given by Bott [2] and Husseini [3].

Our result can be stated as follows;

Theorem 3.1. *Let $U(t)$ be the unitary group and μ the generator* *of* $\pi_1(U(t)) \cong \boldsymbol{Z}$. *Then the iterated operator*

$$D_\mu^s : \pi_r(U(t)) \longrightarrow \pi_{r+s}(U(t))$$

is trivial for any r, *where* (i) $s=2$ *for* t *odd*, (ii) $s=5$ *for* $t \equiv 0$ mod 8 *and* (iii) $s=4$ *otherwise.*

Of course D_μ is trivial when $t=1$. We can show that $D_\mu^3 : \pi_{2t}(U(t)) \longrightarrow \pi_{2t+3}(U(t))$ is non-trivial for $t \equiv 0$ mod 8. But the author does not know of any example that D_μ^4 is non-trivial.

We also give an analogous result for the symplectic group $Sp(t)$ and the generator of $\pi_3(Sp(t))$ in §3. In §4 and §5, we study the relative Samelson product.

2. Preliminary. Let F denote the field of complex numbers C or the field of quaternions H. By F^n we denote an n-dimensional right vector space over F with a fixed basis and the usual inner product. The group of automorphisms of F^n which preserve this inner product is denoted by $G_n = G(F^n)$. As usual, $G(C^n) = U(n)$, $G(H^n) = Sp(n)$. Let $G_{n,k}$ be one of the complex Stiefel manifold $W_{n,k} = U(n)/U(n-k)$ or the symplectic Stiefel manifold $X_{n,k} = Sp(n)/Sp(n-k)$, depending on the field F. We denote the natural projection by $p: G_n \longrightarrow G_{n,k}$. The inclusion $i: G_n \longrightarrow G_{n+m}$ is the mapping which leaves the last m basic vectors fixed.

Let $k \geqq 1$ and m, $n \geqq k$. The intrinsic join operation, as defined in [4], is a pairing of $\pi_i(G_{n,k})$ with $\pi_j(G_{m,k})$ to $\pi_{i+j+1}(G_{n+m,k})$, where $i, j \geqq 0$. The intrinsic join of $\alpha \in \pi_i(G_{n,k})$ with $\beta \in \pi_j(G_{m,k})$ is denoted by $\alpha * \beta$. This pairing is both biliear and associative. Consider the case $k=1$. The $G_{n,1}$ is homeomorphic to S^{dn-1}, where d is the dimension of F over the real field.

Lemma 2.1. *The homomorphism from* $\pi_i(G_{m,1})$ *to* $\pi_{i+dn}(G_{n+m,1})$ *defined by taking the intrinsic join with the generator of* $\pi_{dn-1}(G_{n,1})$ *is essentially the same as the dn-fold iterated suspension homomorphism.*

Bott [2] and Husseini [3] obtained a useful relation between the intrinsic join and the Samelson product:

Theorem 2.2. *For any* $\alpha \in \pi_i(G_n)$ *and* $\beta \in \pi_j(G_m)$,

$$\langle i_*\alpha, i_*\beta\rangle = \pm\partial((p_*\alpha) * (p_*\beta)),$$

as shown in the following diagram;

$$\begin{array}{ccc} & \pi_i(G_n)\times\pi_j(G_m) & \\ {\scriptstyle p_*\times p_*}\swarrow & & \searrow{\scriptstyle i_*\times i_*} \\ \pi_i(G_{n,k})\times\pi_j(G_{m,k}) & & \pi_i(G_{n+m-k})\times\pi_j(G_{n+m-k}) \\ \downarrow * & & \downarrow\langle\ ,\ \rangle \\ \pi_{i-j+1}(G_{n+m,k}) & \xrightarrow{\ \partial\ } & \pi_{i+j}(G_{n+m-k}) \end{array}$$

where ∂ denotes the boundary homomorphism in the homotopy exact sequence of the fiber space $p: G_{n+m} \longrightarrow G_{n+m,k}$ with fiber G_{n+m-k}.

3. The proof of the main Theorem.

Theorem 3.1. *Let $U(m)$ be the unitary group and μ be the generator of $\pi_1(U(m)) \cong \boldsymbol{Z}$. Then the iterated operator*

$$\boldsymbol{D}_\mu^s : \pi_r(U(m)) \longrightarrow \pi_{r+s}(U(m))$$

is trivial for any r, where (i) $s=2$ for m odd, (ii) $s=5$ for $m \equiv 0$ mod 8 and (iii) $s=4$ otherwise.

Proof. Let $\mu' \in \pi_1(U(1))$ be a generator which satisfies $\mu = i_*\mu'$ for the inclusion $i: U(1) \longrightarrow U(m)$. Apply Theorem 2.2 for $\mu' \in \pi_1(U(1))$ and any $\beta \in \pi_r(U(m))$. Since $U(1)$ is identified with $W_{1,1}$, we have from Lemma 2.1

$$\begin{aligned}\boldsymbol{D}_\mu(\beta) &= \langle i_*\mu', \beta\rangle \\ &= \pm\partial E^2(p_*\beta) \in \pi_{r+1}(U(m)),\end{aligned}$$

where E is the suspension homomorphism. Hence

$$\begin{aligned}\boldsymbol{D}_\mu^2(\beta) &= \langle \mu, \langle \mu, \beta\rangle\rangle = \pm \boldsymbol{D}_\mu(\partial E^2(p_*\beta)) \\ &= \pm\partial E^2(p_*\partial E^2(p_*\beta)).\end{aligned}$$

Now the composition

$$p_*\partial : \pi_{r+2}(S^{2m+1}) \longrightarrow \pi_{r+1}(U(m)) \longrightarrow \pi_{r+1}(S^{2m-1})$$

is the boundary homomorphism $\varDelta$ in the exact sequence of the fiber space $W_{m+1,2}$ over $W_{m+1,1} = S^{2m+1}$ with fiber $W_{m,1} = S^{2m-1}$. Then, as well known results,

$$\varDelta(\iota_{2m+1}) = \begin{cases} 0 & \text{for } m \text{ odd} \\ \eta_{2m-1} & \text{for } m \text{ even} \end{cases}$$

where $\pi_{2m}(S^{2m-1}) = \{\eta_{2m-1}\} \cong \boldsymbol{Z}_2$ and $\pi_{2m+1}(S^{2m+1}) = \{\iota_{2m+1}\} \cong \boldsymbol{Z}$. Therefore we have

$$\begin{aligned} D_\mu^2(\beta) &= \pm\partial E^2(\Delta(E^2(p_*\beta))) \\ &= \pm\partial E^2(\Delta(\iota_{2m+1})\circ E(p_*\beta)) \\ &= \begin{cases} 0 & \text{for } m \text{ odd} \\ \partial(\eta_{2m-1}\circ E^3(p_*\beta)) & \text{for } m \text{ even} \end{cases} \end{aligned}$$

Then, after two more steps, we obtain

$$\begin{aligned} D_\mu^4(\beta) &= \partial(\eta_{2m+1}^3\circ E^5(p_*\beta)) \\ &= (\partial\eta_{2m+1}^3)\circ(E^4(p_*\beta)) \end{aligned}$$

for m even.

By Matsunaga [8], $\pi_{2m-3}(U(m))$ is generated by $\partial\nu_{2m+1}$ and its 2-primary components

$$^2\pi_{2m-3}(U(m)) \cong \begin{cases} Z_2 & \text{if } m \equiv 2 \bmod 4 \\ Z_4 & \text{if } m \equiv 4 \bmod 8 \\ Z_8 & \text{if } m \equiv 0 \bmod 8, \end{cases}$$

where $\pi_{2m+4}(S^{2m+1}) = \{\nu_{2m+1}\} \cong Z_{24}$ $(m \geqq 2)$. Since $\eta_{2m+1}^3 = 12\nu_{2m+1}$ it follows that

$$D_\mu^4(\beta) = 0$$

for $m \equiv 2,4,6 \bmod 8$. After one more step, we have

$$D_\mu^5(\beta) = (\partial\eta_{2m+1}^4)\circ E^5(p_*\beta).$$

Thus we have

$$D_\mu^5(\beta) = 0$$

for $m \equiv 0 \bmod 8$, since $\eta_{2m+1}^4 = 0$. Q.E.D.

Example. Let α_m be a generator of $\pi_{2m}(U(m)) \cong Z_{m!}$. Specifically, we shall take α_m to be $\alpha_m = \partial\iota_{2m+1}$. Then we have

$$\begin{aligned} D_\mu(\alpha_{2m}) &= \alpha_{2m}\circ\eta_{4m} \neq 0 \\ D_\mu^2(\alpha_{2m}) &= \alpha_{2m}\circ\eta_{4m}^2 \neq 0 \\ D_\mu^3(\alpha_{2m}) &= \alpha_{2m}\circ\eta_{4m}^3 \qquad (\text{If } m \equiv 0 \bmod 4 \text{ then } \alpha_{2m}\circ\eta_{4m}^3 \neq 0) \end{aligned}$$

and

$$D_\mu^4(\alpha_{2m}) = D_\mu(\alpha_{2m+1}) = 0.$$

Let

$$D_\tau : \pi_r(Sp(m)) \longrightarrow \pi_{r+3}(Sp(m))$$

be the operator defined by taking the Samelson product with the generator $\tau \in \pi_3(Sp(m)) \cong Z$. Then we have

Theorem 3.2. *The iterated operator*

$$D_\tau^s : \pi_r(Sp(m)) \longrightarrow \pi_{r+3s}(Sp(m))$$

of D_τ *is trivial for any* r, *where* (i) $s=2$ *if* $m \equiv -1 \bmod 24$, (ii) $s=3$ *if* m *is other odd and* (iii) $s=5$ *if* m *is even.*

Proof. $Sp(1)$ and $X_{1,1}$ are identified with S^3. For the generator τ' of $\pi_3(Sp(1))$, the generator τ of $\pi_3(Sp(m))$ satisfies $\tau = i_*\tau'$ for the inclusion $i: Sp(1) \longrightarrow Sp(m)$. From Theorem 2.2 and Lemma 2.1, it follows that

$$D_\tau(\beta) = \langle i_*\tau', \beta\rangle = \pm \partial E^4(p_*\beta) \tag{3.3}$$

for any $\beta \in \pi_r(Sp(m))$.

Now the composition

$$p_*\partial : \pi_{r+4}(S^{4m+3}) \longrightarrow \pi_{r+3}(Sp(m)) \longrightarrow \pi_{r+3}(S^{4m-1})$$

is the boundary homomorphism Δ in the exact sequence of the fiber space $X_{m+1,2}$ over $X_{m+1,1} = S^{4m+3}$ with fiber $X_{m,1} = S^{4m-1}$.

Then we have

$$\Delta(\iota_{4m+3}) = \begin{cases} \omega & \text{for } m=1 \\ (m+1)\nu_{4m-1} & \text{for } m \geqq 2 \end{cases}$$

where $\pi_6(S^3) = \{\omega\} \cong Z_{12}$. Thus we obtain

$$\begin{aligned} D_\tau^2(\beta) &= \pm \partial E^4(p_*\partial E^4(p_*\beta)) \\ &= \begin{cases} \pm \partial((E^4\omega)\circ(E^7 p_*\beta)) & \text{for } m=1 \\ \pm \partial((m+1)\nu_{4m+3}\circ(E^7 p_*\beta)) & \text{for } m \geqq 2. \end{cases} \end{aligned}$$

Thus $D_\tau^2(\beta) = 0$ for $m+1 \equiv 0 \bmod 24$.

On iterating (3.3), we get

$$D_\tau^3(\beta) = \pm \partial((m+1)^2 \nu_{4m+3}^2 \circ E^{10} p_*\beta). \tag{3.4}$$

Since $2\nu_{4m+3}^2 = 0$, it follows from (3.4) that $D_\tau^3(\beta) = 0$ for m odd. After two more steps, we have $D_\tau^5(\beta) = 0$, since $\nu_{4m+3}^4 = 0$. Q.E.D.

Corollary 3.5 (Arkowitz-Curjel [1]). *If* $1 \in \pi_3(S^3)$ *is the homotopy class of the identity map, then* $\langle\langle\langle 1,1\rangle,1\rangle,1\rangle = 0 \in \pi_{12}(S^3)$.

4. The relative Samelson product. The definition and the material in this section are due to James [6]. Let H be a subgroup of the topological group G. The relative Samelson product $\langle\ ,\ \rangle$ is a pairing

$$\pi_p(H) \times \pi_q(G,H) \longrightarrow \pi_{p+q}(G,H) \qquad (p \geqq 1,\ q \geqq 2)$$

defined as follows. Let $\alpha \in \pi_p(H)$, $\beta \in \pi_q(G,H)$ be represented by maps

$$f : (I^p, \dot{I}^p) \longrightarrow (H,e), \quad g : (I^q, \dot{I}^q) \longrightarrow (G,H).$$

Then $\langle \alpha, \beta \rangle \in \pi_{p+q}(G,H)$ is represented by the map

$$k : (I^p \times I^q, I^p \times \dot{I}^q \cup \dot{I}^p \times I^q) \longrightarrow (G,H),$$

where $k(x,y) = f(x)g(y)f(x)^{-1}g(y)^{-1}$ for $x \in I^p$, $y \in I^q$.

The main relations between the ordinary and relative Samelson product are indicated in the following diagram:

(4.1)
$$\begin{array}{ccc} \pi_p(H) \times \pi_q(G,H) & \xrightarrow{\langle\ ,\ \rangle} & \pi_{p+q}(G,H) \\ \downarrow 1 \times \partial & & \downarrow \partial \\ \pi_p(H) \times \pi_{q-1}(H) & \xrightarrow{\langle\ ,\ \rangle} & \pi_{p+q-1}(H) \end{array}$$

(4.2)
$$\begin{array}{ccccc} & & \pi_p(H) \times \pi_q(G) & & \\ & \swarrow i_* \times 1 & & 1 \times j_* \searrow & \\ \pi_p(G) \times \pi_q(G) & & & & \pi_p(H) \times \pi_q(G,H) \\ \downarrow \langle\ ,\ \rangle & & & & \downarrow \langle\ ,\ \rangle \\ \pi_{p+q}(G) & & \xrightarrow{j_*} & & \pi_{p+q}(G,H) \end{array}$$

The homomorphism i_*, j_*, ∂, of course, are from the homoyopy exact sequence of the pair (G,H) and the diagrams are commutative up to sign. We see from this that an element $\gamma \in \pi_r(H)$ determines a homomorphism of the homotopy exact sequence into itself raising dimension by r. On $\pi_*(H)$ we take the ordinary Samelson product with γ itself, on $\pi_*(G)$ the ordinary Samelson product with $i_*(\gamma)$, and on $\pi_*(G,H)$ the relative Samelson product with γ itself. And we denote by $\boldsymbol{D}_{H,\gamma}$, $\boldsymbol{D}_{G,\gamma}$ and $\boldsymbol{D}_{G/H,\gamma}$ each Samelson product respectively.

Lemma 4.3. *If $\boldsymbol{D}^s_{H,\gamma} = 0$ and $\boldsymbol{D}^t_{G,\gamma} = 0$, then $\boldsymbol{D}^{s+t}_{G/H,\gamma} = 0$.*

Proof. If $\alpha \in \pi_p(G,H)$, then $\partial \boldsymbol{D}^s_{G/H,\gamma}(\alpha) = \pm \boldsymbol{D}^s_{H,\gamma}(\partial\alpha) = 0$ by (4.1). Hence $\boldsymbol{D}^s_{G/H,\gamma}(\alpha) = j_*(\varepsilon)$, by exactness, for some $\varepsilon \in \pi_{p+rs}(G)$. Thus, from (4.2), $\boldsymbol{D}^{s+t}_{G/H,\gamma}(\alpha) = \boldsymbol{D}^t_{G/H,\gamma}(j_*\varepsilon) = \pm j_* \boldsymbol{D}^t_{G,\gamma}(\varepsilon) = 0$. Q.E.D.

Proposition 4.4. *For the pair (G,H) and $\gamma \in \pi_r(H)$ in the following table, there exists an integer s for which the s-fold iterated operator $\boldsymbol{D}^s_{G/H,\gamma}$ of $\boldsymbol{D}_{G/H,\gamma} : \pi_p(G,H) \longrightarrow \pi_{p+r}(G,H)$ is trivial for any $p \geqq 2$;*

	(G,H)	$\gamma \in \pi_r(H)$	s
(i)	$(U(n+k), U(n))$	γ *is a generator of* $\pi_1(U(n))$	10
(ii)	$(R_{2n}, U(n))$	γ *is a generator of* $\pi_1(U(n))$	11
(iii)	$(Sp(n+k), Sp(n))$	γ *is a generator of* $\pi_3(Sp(n))$	10.

Proof. Let γ be a generator of $\pi_r(H)$. Then the image of γ in $\pi_r(G)$ is a generator of $\pi_r(G)$. Thus from Lemma 4.3 and Theorems 3.1, 3.2, we obtain the results. Q.E.D.

5. The relative Samelson product on $(Sp(n), U(n))$. Consider the pair $(Sp(n), U(n))$. By the Bott periodicity, $\pi_{2m-1}(U(n+m))$ is infinite cyclic group for $n \geqq 0$ and $\pi_{2n}(Sp(n+m), U(n+m))$ is infinite cyclic group for $m \geqq 0$ if $n \equiv 1$ or $3 \bmod 4$. The boundary homomorphism $\partial: \pi_{2n}(Sp(n+m), U(n+m)) \longrightarrow \pi_{2n-1}(U(n+m))$ is an isomorphism for $n \equiv 1 \bmod 4$ and maps a generator onto twice a generator for $n \equiv 3 \bmod 4$. For the first non-stable range, we have

Lemma 5.1 (See [7]). *The following sequence*

$$0 \longrightarrow \pi_{2t+1}(Sp(t)) \xrightarrow{j_*} \pi_{2t+1}(Sp(t), U(t)) \xrightarrow{\partial} \pi_{2t}(U(t)) \longrightarrow 0$$

is exact and

$$\pi_{2t+1}(Sp(t), U(t)) = \begin{cases} Z_{t!} & \text{if } t \equiv 0 \bmod 4 \\ Z+Z_2 & \text{if } t \equiv 1 \bmod 4 \\ Z_{2\times t!} & \text{if } t \equiv 2 \bmod 4 \\ Z & \text{if } t \equiv 3 \bmod 4. \end{cases}$$

We apply the diagram (4.1) to the pair $(Sp(n+m), U(n+m))$. Then, from Theorem 1 of Bott [2], we obtain

Proposition 5.2. *Let* $m, n \geqq 1$ *with* $n \equiv 1$ *or* $3 \bmod 4$. *Consider the relative Samelson product*

$$\langle \phi_m, \zeta_n \rangle \in \pi_{2m+2n+1}(Sp(n+m), U(n+m)),$$

where $\phi_m \in \pi_{2m+1}(U(n+m))$ *and* $\zeta_n \in \pi_{2n}(Sp(n+m), U(n+m))$ *are generators. Let* $\xi \in \pi_{2m+2n+1}(Sp(n+m), U(n+m))$ *be a generator such that* $\partial\xi$ *is a generator of* $\pi_{2n+2m}(U(n+m))$, *then*

$$\langle \phi_m, \zeta_n \rangle = \begin{cases} m!(n-1)!\xi & \bmod \textit{image } j_* \textit{ if } n \equiv 1 \bmod 4 \\ 2(m!(n-1)!)\xi & \bmod \textit{image } j_* \textit{ if } n \equiv 3 \bmod 4, \end{cases}$$

where $j_*: \pi_{2n+2m+1}(Sp(n+m)) \longrightarrow \pi_{2n+2m+1}(Sp(n+m), U(n+m))$.

By taking various n and m we obtain examples of non-trivial relative Samelson products in the case of $(Sp(t), U(t))$. Hence we deduce

Corollary 5.3. *If* $t \geqq 2$, *then* $U(t)$ *is not homotopy normal in* $Sp(t)$ *in the sense of McCarty* [9].

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(*Received January 12, 1982*)