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AN APPLICATION OF CERTAIN MULTIPLICITIES OF C^∞ MAP GERMS

Dedicated to Professor K. Murata on his 60th birthday

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Introduction. The differentiable classification theory of C^∞ stable map germs of $(\boldsymbol{R}^n, 0)$ into $(\boldsymbol{R}^p, 0)$ has been developed in [7]. However, very little is known about topological invariants for the topological classification of C^∞ stable map germs in dimensions $n > p$.

In this paper we will define the i-homological multiplicity of a C^∞ map germ for every nonnegative integer i which is a topological invariant for C^∞ map germs. In dimensions $n \leq p$ J. Damon and A. Galligo [3] have defined the real multiplicity for a C^∞ map germ. The 0-homological multiplicity turns out to coincide with the real multiplicity for C^∞ stable map germs. By using the real multiplicity J. Damon has shown that the kernel rank $\mathrm{kr}(f)$ of a C^∞ stable map germ which is the dimension of $\mathrm{Ker}(df)$ at the origin is a topological invariant when $n \leq p$ ([1, Theorem 3]). As an application of the homological multiplicities we will give another proof of the following theorem which is a part of [2, Theorem 2].

Theorem ([2]). *For C^∞ stable map germs $f:(\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^p, 0)$ the kernel rank $\mathrm{kr}(f)$ is a topological invariant in dimensions $n > p$.*

Let f be a C^∞ map of a differentiable manifold N into a differentiable manifold P. Let $S^i(f)$ denote the set of all points x of N such that the dimension of $\mathrm{Ker}(df_x)$ is i.

The following is a consequence of Theorem.

Corollary. *Let $f, g: N \longrightarrow P$ denote proper C^∞ stable maps. If f is topologically equivalent to g, then $S^i(f)$ is homeomorphic to $S^i(g)$.*

We will give preliminaries and notations in Section 1 and define the homological multiplicities of C^∞ map germs in Section 2. In Section 3 we will show by using the normal forms of C^∞ stable map germs that in dimensions $n \geq p$ the homological multiplicities distinguish the topological type of a C^∞ stable map germ with kernel rank $n-p$ from that of a C^∞ stable map germ with kernel rank $n-p+1$. This is the first step of the induction in the proof of the above theorem in Section 4.

All manifolds will be paracompact Hausdorff and C^∞ manifolds.

After the first draft of the paper was written, the author was informed of the reference [2] by S.Izumiya.

1. Preliminaries. We consider C^∞ map germs $f: (N,x) \longrightarrow (P,y)$ which are germs of differentiable maps from a manifold N into a manifold P with $f(x)=y$. Two C^∞ map germs $f:(N,x)\longrightarrow(P,y)$, $g:(N',x')\longrightarrow(P',y')$ are called topologically equivalent if there exist germs of homeomorphisms $h:(N,x)\longrightarrow(N',x')$ and $k:(P,y)\longrightarrow(P',y')$ such that two C^∞ map germs $k\circ f$ and $g\circ h$ are equal. Let C denote a certain class of C^∞ map germs. An invariant associated to a C^∞ map germ is called a topological invariant for C if for any two topologically equivalent C^∞ map germs of C, their invariants are equal.

Let x be a point of a manifold N. Let X and Y be subsets of N which are contained in neighbourhoods U and V of x respectively. Then X and Y are called equivalent at the point x when there exists a neighbourhood W of x such that $W\subset U\cap V$ and $X\cap W=Y\cap W$. For a subset X and a point x of N, we define the set germ of X at a point x, denoted by X_x, as the equivalence class of a subset X at x. We consider another set germ Y_y for a manifold M, its subset Y and a point y. We will also say that two set germs X_x and Y_y are topologically equivalent if there is a germ of a homeomorphism of N into M with source x and target y such that X is mapped onto Y near x.

Next we give an example of a set germ associated to a C^∞ map germ $f:(N,x)\longrightarrow(P,y)$. Let $f':U\longrightarrow P$ be a representative of f. We define $S^i(f')$ to be the set of all points a of U such that the dimension of $\mathrm{Ker}(df'_a)$ is i. It is clear that the set germ $S^i(f')_x$ depends only on the C^∞ map germ f. Therefore we denote it by $S^i(f)_x$. Let $J^1(N,P)$ denote the space of 1-jets. Let $\Sigma^i(N,P)$ denote the set of all elements z of $J^1(N,P)$ whose kernel rank of a representative map germ is i (see [6]). Then by the definition of $S^i(f')$ we have

$$S^i(f') = (j^1 f')^{-1}(\Sigma^i(U,P)).$$

We will say that a C^∞ map germ $f:(N,x)\longrightarrow(P,y)$ is transverse to $\Sigma^i(N,P)$ at a point x if a representative 1-jet $j^1(f')$ is transverse to $\Sigma^i(U,P)$ at a point x. This definition does not depend on a choice of a representative f'. By a standard argument in differential topology we have the following lemma.

Lemma 1.1. *Let $f:(N,x)\longrightarrow(P,y)$ be a C^∞ map germ whose kernel* *rank at x is i. If f is transverse to $\Sigma^i(N,P)$ at x, then the set germ $S^j(f)_x$ is not empty for* $\max(0, n-p) \leq j \leq i$.

Remark 1.2. We remark that a C^∞ stable map germ $f:(N,x) \longrightarrow (P,y)$ is transverse to $\Sigma^i(N,P)$ at x. This will be shown easily, for example, by [7, Proposition 1.8].

Next we recall the construction of the normal forms of C^∞ stable map germs in [7]. In the paper we will only need the normal forms of C^∞ stable map germs with kernel rank $n-p+1$ for $n \geq p$. Let $\boldsymbol{R}[[x_p,\cdots,x_n]]$ denote the ring of formal power series in indeterminates $x_p,\cdots,x_n$ and let $\mathfrak{m}$ denote its unique maximal ideal. Let q be a polynomial in $x_p,\cdots,x_n$. Let $\psi(q)$ denote the ideal of $\mathfrak{m}$ generated by $\partial q/\partial x_p,\cdots,\partial q/\partial x_n$ and q. We put

$$c = c(q) = \dim(\mathfrak{m}/\psi(q)).$$

Then we can choose a set of c elements of $\mathfrak{m}$, $v_1,\cdots,v_c$ such that their canonical images form a basis of the vector space $\mathfrak{m}/\psi(q)$. Suppose that $c \leq r$ and every v_i is a polynomial. Define a C^∞ map germ $f:(\boldsymbol{R}^n,0) \longrightarrow (\boldsymbol{R}^p,0)$ with $f=(f_1,\cdots,f_p)$ as follows.

$$(*) \qquad f_i = \begin{cases} x_i & (0 \leq i \leq p-1) \\ q+\sum_{j=1}^{c} x_j v_j & (i=p). \end{cases}$$

Then f becomes a C^∞ stable map germ and every C^∞ stable map germ with rank $p-1$ at the origin is written as above (see [7, Theorem 5.10]).

2. The homological multiplicities. Let $f:(\boldsymbol{R}^n,0) \longrightarrow (\boldsymbol{R}^p,0)$ be a C^∞ map germ. Let $f':U \longrightarrow \boldsymbol{R}^p$ be a representative of f defined in a neighbourhood U of the origin. For any neighbourhood V of the origin in U we define $m_i(f',V)$ to be the maximum of all ranks of $H_i((f')^{-1}(a) \cap V; \boldsymbol{Z})$ where a is a point of $\boldsymbol{R}^p$. If no such maximum exists, then we put $m_i(f',V)=\infty$. We define $m_i(f')$ to be the minimum of all $m_i(f',V)$ where V is any neighbourhood of the origin in U. Similarly $m_i(f')$ may be ∞. Consider the restriction map $f'|_V: V \longrightarrow \boldsymbol{R}^p$ for a neighbourhood V of the origin in U. Then we have

$$m_i(f'|_V) \geq m_i(f').$$

Definition 2.1. For a C^∞ map germ $f:(\boldsymbol{R}^n,0) \longrightarrow (\boldsymbol{R}^p,0)$ we define the *i-homological multiplicity* $m_i(f)$ as

$$m_i(f) = \max\{m_i(f') \mid f' \text{ is a representative of } f\}$$

Remark 2.2. J.Damon and A.Galligo [3] have also defined the real multiplicity $m(f)$ of a C^∞ map germ in dimensions $n \leq p$ by considering the number of points of $(f')^{-1}(a) \cap V$. If $n \leq p$ and the real multiplicity is finite, then the 0-homological multiplicity $m_0(f)$ coincides with $m(f)$. Especially both of multiplicities are equal for a C^∞ stable map germ, since it is shown in [3, Proposition 2.4] that $m(f)$ is finite for a C^∞ stable map germ f. However it is easily seen that those multiplicities are not equal in general.

Lemma 2.3. *The i-homological multiplicity $m_i(f)$ is a topological invariant for C^∞ map germs f.*

Proof. Let two C^∞ map germs f and g be topologically equivalent. Then there exist their representatives $f' : U \longrightarrow W$ and $g' : U' \longrightarrow W'$ homeomorphisms $h : U \longrightarrow U'$ and $k : W \longrightarrow W'$ such that the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{\;f'\;} & W \\ \Big\downarrow h & & \Big\downarrow k \\ U' & \xrightarrow{\;g'\;} & W' \end{array}$$

Then for any $V \subset U$ and $y \in W$ we have

$$h((f')^{-1}(y) \cap V) = (g')^{-1}(k(y)) \cap h(V).$$

This shows that $m_i(f') = m_i(g')$. Hence we have $m_i(f) = m_i(g)$.

Lemma 2.4. *Let $f : (\boldsymbol{R}^n, 0) \longrightarrow (\boldsymbol{R}^p, 0)$ be a C^∞ map germ with rank* $\min(n,p)$. *Then*

$$m_i(f) = \begin{cases} 1 & \textit{if } i = 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $f' : U \longrightarrow \boldsymbol{R}_p$ be any representative of f. Then there exists a neighbourhood V of the origin in U where there exist local coordinates $x_1, \cdots, x_n$, null at 0 and $y_1, \cdots, y_p$, null at 0 such that

$$\begin{cases} y_i \circ f = x_i, & 1 \leq i \leq \min(n,p) \\ y_i \circ f = 0, & n+1 \leq i \leq p \text{ when } n < p. \end{cases}$$

First we suppose that $n > p$. Then for any neighbourhood W of the origin in V, we can take an open ε-ball $\mathring{D}_\varepsilon$ with center 0 in W such that $(f')^{-1}(a) \cap \mathring{D}_\varepsilon$ is either diffeomorphic to an open ball of dimension $n-p$

or empty. Therefore we have that $m_0(f' \mid \dot{D}_\varepsilon) = 1$ and $m_i(f' \mid \dot{D}_\varepsilon) = 0$ for $i > 0$. On the other hand we have $m_i(f' \mid \dot{D}_\varepsilon) \geq m_i(f')$. Since $m_i(f)$ is the maximum of all $m_i(f')$ where f' is a representative of f, we obtain that $m_0(f) = 1$ and $m_i(f) = 0$ for $i > 0$. The proof for the case $p \geq n$ is similar.

3. C^∞ stable map germs and homological multiplicities. In this section we will show that the homological multiplicities distinguish the topological type of a C^∞ map germ with rank p from that of a C^∞ stable map germ with rank $p-1$ when $n \geq p$.

Lemma 3.1. *Let $f:(\boldsymbol{R}^n,0) \longrightarrow (\boldsymbol{R},0)$ be the C^∞ function germ given by $x_1^2+ \cdots +x_r^2-(x_{r-1}^2+ \cdots +x_n^2)$. Then*

$$m_{r-1}(f) = \begin{cases} 1 & \text{if } r > 1 \\ 2 & \text{if } r = 1 \end{cases}$$

and

$$m_{n-r-1}(f) = \begin{cases} 1 & \text{if } n-r > 1 \\ 2 & \text{if } n-r = 1. \end{cases}$$

Proof. For $\varepsilon > 0$ and $\delta > 0$, we let

$$V_\varepsilon = \{(x_1,\cdots,x_r) \mid x_1^2+ \cdots +x_r^2 < \varepsilon^2\}$$
$$U_\delta = \{(x_{r+1},\cdots,x_n) \mid x_{r+1}^2+ \cdots +x_n^2 < \delta^2\}.$$

We consider the function $x_1^2+ \cdots +x_r^2-(x_{r-1}^2+ \cdots +x_n^2)$ on $V_\varepsilon \times U_\delta$ which is denoted by f'. Then for any a with $0< a<\varepsilon$, $(f')^{-1}(a)$ is diffeomorphic to $S^{r-1} \times \mathring{D}^{n-r}$ by the following map

$$p: s^{r-1} \times \mathring{D}^{n-r} \longrightarrow V_\varepsilon \times U_\delta$$

defined by

$$p((x_1,\cdots,x_r), (x_{r+1},\cdots,x_n)) = (\sqrt{a+b^2(x_{r+1}^2+ \cdots +x_n^2)}(x_1,\cdots,x_r), b(x_{r+1},\cdots,x_n))$$

where b is $\min(\delta, \sqrt{\varepsilon^2-a^2})$. Let W be any neighbourhood of the origin in $V_\varepsilon \times U_\delta$. Then there exist $\varepsilon' > 0$ and $\delta' > 0$ such that $V_{\varepsilon'} \times U_{\delta'} \subset W$. For any a' with $0 < a' < \varepsilon'$ we have

$$H_{r-1}((f')^{-1}(a') \cap (V_{\varepsilon'} \times U_{\delta'}); \boldsymbol{Z}) = \begin{cases} \boldsymbol{Z} & \text{if } r > 1 \\ \boldsymbol{Z} \oplus \boldsymbol{Z} & \text{if } r = 1 \end{cases}$$

and the inclusion

$$i: (f')^{-1} \cap (V_{\varepsilon'} \times U_{\delta'}) \longrightarrow (f')^{-1}(a') \cap W$$

induces the injective homomorphism

$$i_*: H_{r-1}((f')^{-1}(a') \cap (V_{\varepsilon'} \times U_{\delta}); \boldsymbol{Z}) \longrightarrow H_{r-1}((f')^{-1}(a') \cap W; \boldsymbol{Z}).$$

We can also show that $(f')^{-1}(a') \cap (V_{\varepsilon'} \times U_{\delta})$ is empty when $a' > \varepsilon'$ and homeomorphic to S^{r-1} when $a' = \varepsilon'$. If $a' = 0$, then it is contractible.

For a negative real number a, we have similarly that $(f')^{-1}(a')$ is diffeomorphic to $S^{n-r-1} \times \mathring{D}^r$ when $0 < |a| < \delta$, to S^{n-r-1} when $|a| = \delta$ and empty otherwise. Thus we have shown that for any representative f' there exists a restriction f'' with $m_{r-1}(f'') = 1$ if $r > 1$ and $m_{r-1}(f'') = 2$ if $r = 1$. This shows the lemma for $m_{r-1}(f)$. The case of $m_{n-r-1}(f)$ can be shown similarly.

Lemma 3.2. *Let U be a neighbourhood of the origin of $\boldsymbol{R}^{n+1}$ and $f': U \longrightarrow \boldsymbol{R}^2$ be the C^∞ map given by*

$$f'(x,\cdots,x_{n+1}) = (x_{n+1},\ x_1^2 + \cdots + x_s^2 - x_{s+1}^2 - \cdots - x_{n-1}^2 + x_n^3 + x_{n+1}x_n).$$

Then

$$m_s(f') \geq 1,\ m_{n-s-1}(f') \geq 1 \ \textit{and}\ m_0(f') = 3 \ \textit{if}\ n = 1.$$

Proof. We consider the range of (a,b) where the equation $x_n^3 + ax_n + b = 0$ has three real roots $r_1(a,b)$, $r_2(a,b)$ and $r_3(a,b)$ with $r_1(a,b) \leq r_2(a,b) \leq r_3(a,b)$. Let V be any neighbourhood of the origin U. We will show that there exists such a that the ranks of $H_s((f')^{-1}((a,0) \cap V; \boldsymbol{Z})$ and $H_{n-s-1}((f')^{-1}((a,0)) \cap V; \boldsymbol{Z})$ are greater than 0. Then we have

$$m_s(f', V) \geq 1 \text{ and } m_{n-s-1}(f', V) \geq 1,$$

which is what we want by the definition of $m_i(f')$. In fact, if we take a sufficiently small negative number a, then we can show that the set consisting of $(x_1, \cdots, x_{n+1}) \in V$ such that $x_{s+1} = \cdots = x_{n-1} = 0$, $x_{n+1} = a$ and x_n is either $r_2(a,\ x_1^2 + \cdots + x_s^2)$ or $r_3(a,\ x_1^2 + \cdots + x_s^2)$ is homeomorphic to S^s. It is easily seen that the inclusion map

$$i: S^s \longrightarrow (f')^{-1}((a,0)) \cap V$$

induces the injective homomorphism

$$i_*: H_s(S^s; \boldsymbol{Z}) \longrightarrow H_s((f')^{-1}((a,0)) \cap V; \boldsymbol{Z}).$$

Hence we have $m_s(f') \geq 1$.

For $m_{n-s-1}(f')$ we consider similarly the set consisting of $(x_1, \cdots, x_{n+1}) \in V$ such that $x_1 = \cdots = x_s = 0$, $x_{n+1} = a$ and x_n is either $r_1(a,\ -x_{s+1}^2 - \cdots - x_{n-1}^2)$ or $r_2(a,\ -x_{s+1}^2 - \cdots - x_{n-1}^2)$. Then we have that it is homeomorphic to S^{n-s-1} and that the inclusion map

$$i : S^{n-s-1} \longrightarrow (f')^{-1}((a,0)) \cap V$$

induces the injective homomorphism

$$i_* : H_{n-s-1}(S^{n-s-1} ; \boldsymbol{Z}) \longrightarrow H_{n-s-1}((f')^{-1}((a,0)) \cap V ; \boldsymbol{Z}).$$

Thus we have $m_{n-s-1}(f') \geq 1$. The proof for the case $n = 1$ is clear.

Proposition 3.4. *Let f, $g : (\boldsymbol{R}^n,0) \longrightarrow (\boldsymbol{R}^p,0)$ $(n \geq p)$ denote C^∞ stable map germs with ranks $p-1$ and p respectively. Then f is not topologically equivalent to g.*

Proof. We will show that there exists an integer i such that $m_i(f)$ and $m_i(g)$ are different. The homological multiplicity $m_i(g)$ is given in Lemma 2.4. We can choose a representative $f' : U \longrightarrow \boldsymbol{R}^p$ of f which is written as (*) in Section 1. First we remark that $(f')^{-1}(a_1,\cdots,a_p)$ is homeomorphic to $(f'_p)^{-1}(a_p) \cap U_a$ where U_a denotes the subset of U with $x_1 = a_1,\cdots,x_{p-1} = a_{p-1}$. Let r denote the rank of the Hessian of $q(x_p,\cdots,x_n)$. Then by the splitting lemma (see, for example, [4,(4.3)]) we may suppose that in a neighbourhood V of the origin in U, q is written as follows

$$q(x_p,\cdots,x_n) = x_p{}^2 + \cdots + x_{p+s}^2 - x_{p+s+1}^2 - \cdots - x_{p+r-1}^2 + q'(x_{p+r},\cdots,x_n),$$

where $s \leq r-1$, $q' \in m'^3$ and m' is the ideal generated by $x_{p+r},\cdots,x_n$.

If $r = n-p+1$, then the number c is zero. It follows from Lemma 3.1 and the above remark that

$$m_s(f) = \begin{cases} 1 & \text{if } s > 0 \\ 2 & \text{if } s = 0 \end{cases}$$

$$m_{n-p-s-1}(f) = \begin{cases} 1 & \text{if } n-p-s > 1 \\ 2 & \text{if } n-p-s = 1. \end{cases}$$

Next we consider the case of $r < n-p+1$. Then the module $m/\psi(q)$ is isomorphic to $m'/\psi(q')$ where $\psi(q')$ is the ideal generated by q' and $\partial q'/\partial x_{p+r},\cdots,\partial q'/\partial x_n$. Then we may choose a set of c elements $v_1,\cdots,v_c$ in (*) of Section 1 such that $v_1,\cdots,v_t$ are all monomials of degree 2 in m' which span m'^2 modulo m'^3 together with $\partial q'/\partial x_{p+r},\cdots,\partial q'/\partial x_n$ and $v_{t+1},\cdots,v_c$ are other elements of m'. Then the following two cases are possible. The rank of the Hessian of $a_1v_1+a_2v_2+\cdots+a_tv_t$ is generically either $n-p-r+1$ or $n-p-r$ for $(a,\cdots,a_t)$.

If the rank of the Hessian of $a_1v_1+\cdots+a_tv_t$ is generically $n-p-r+1$ for $(a_1,\cdots,a_t)$, then in a neighbourhood V of the origin in U the polynomial

$$q'+\sum_{i=1}^{t} a_i v_i$$

is written as

$$x_{p+r}^2+\cdots+x_{p+r+u}^2-x_{p+r+u+1}^2-\cdots-x_n^2$$

under a change of coordinate of $(x_{p+r},\cdots,x_n)$. Hence it follows from Lemma 3.1 and the above remark that

$$m_{s+u+1}(f)\geq d \quad \text{where} \begin{cases} d=1 & \text{for } s+u+1>0 \\ d=2 & \text{for } s+u+1=0 \end{cases}$$

and

$$m_{n-p-s-u-2}(f)\geq d' \quad \text{where} \begin{cases} d'=1 & \text{for } n-p-s-u>2 \\ d'=2 & \text{otherwise.} \end{cases}$$

Now consider the other case. Then q' is not an element of m'^4. Hence we may suppose by the well known result (see [4,(4.6)]) that $q'+a_1v_1+\cdots+a_tv_t$ is written as

$$x_{p+r}^2+\cdots+x_{p+r+u}^2-x_{p+r+u+1}^2-\cdots-x_{n-1}^2+x_n^3$$

under a change of coordinate of $x_{p+r},\cdots,x_n$. Moreover we may take x_n as v_{t+1} in (*) of Section 1. Now we consider f_{p+1} for $a_{t+2}=\cdots=a_p=0$ under the coordinate of $\boldsymbol{R}^n$ which is written as

$$x_p^2+\cdots+x_{p+s}^2-x_{p+s+1}^2-\cdots-x_{p+r-1}^2+q'+a_1v_1+\cdots+a_tv_t+a_{t+1}x_n.$$

Then by Lemma 3.2 we can verify that

$$m_{s+u+2}(f')\geq 1 \text{ and } m_{n-s-u-p-2}(f')\geq 1.$$

Therefore we have shown the proposition by Lemma 2.3 and Lemma 2.4.

4. A proof of Theorem. In this section we give a sketch of the proof of Theorem, since Proposition 3.4 enables us to follow the proof of [1, Theorem 3].

Let Σ^i be the subset of $\mathrm{Hom}(\boldsymbol{R}^n, \boldsymbol{R}^p)$ of the linear homomorphisms with kernel rank i. We have a bundle homomorphism of constant rank $n-i$, $h:\Sigma^i\times\boldsymbol{R}^n\longrightarrow\Sigma^i\times\boldsymbol{R}^p$ defined by

$$h((f,v))=(f,f(v)).$$

Let K and C denote the bundles $\mathrm{Ker}(h)$ and $\mathrm{Cok}(h)$ respectively. Then there is a map $\phi:\mathrm{Hom}(K,C)\longrightarrow\mathrm{Hom}(\boldsymbol{R}^n,\boldsymbol{R}^p)$ given by

$$\phi((f,g))=f+g$$

where f is an element of Σ^i and g an element of $\mathrm{Hom}(K,C)$ over f. We here identify g with an element of $\mathrm{Hom}(\boldsymbol{R}^n,\boldsymbol{R}^p)$ by the composition map

$$\boldsymbol{R}^n \xrightarrow{\text{projection}} K \xrightarrow{g} C \subset \boldsymbol{R}^p.$$

We can show that ϕ gives a diffeomorphism of a small tubular neighbourhood Z of the zero-section of $\mathrm{Hom}(K,C)$ onto a small neighbourhood of Σ^i in $\mathrm{Hom}(\boldsymbol{R}^n,\boldsymbol{R}^p)$.

Let $S(K)$ and $S(C)$ denote the associated sphere bundles of K and C respectively. We consider a $\boldsymbol{Z}_2$-action on $S(K)\times S(C)$ by mapping (v,v') onto $(-v,-v')$ where $v\in S(K)$ and $v'\in S(C)$ and denote its quotient space by $S(K)\times_{Z_2}S(C)$. Let $C(S(K)\times_{Z_2}S(C))$ be the open mapping cylinder of $S(K)\times_{Z_2}S(C)$ over Σ^i. Let $i>\max(0,\ n-p)$. Then $(\Sigma^{i-1}\cup\Sigma^i)\cap\phi(Z)$ is homeomorphic to $C(S(K)\times_{Z_2}S(C))$.

Proof of Theorem. The case of $n\leq p$ has already been treated in [1]. So we suppose that $n\geq p$. We prove the following statement by induction on i: Let f be a C^∞ stable map germ of kernel rank i. Then f is not topologically equivalent to any C^∞ stable map germ of kernel rank j with $j<i$.

Let $i=n-p+1$. Then the statement follows from Proposition 3.4. Next we suppose that the above statement is true for every i with $i<k$ $(k>n-p+1)$. Soppose that a C^∞ stable map germ f of kernel rank k is topologically equivalent to a C^∞ stable map germ g of kernel rank j with $j<k$. Then there exist representatives $f':U_1\longrightarrow V_1$ and $g':U_2\longrightarrow V_2$ of f and g respectively and homeomorphisms $h:U_1\longrightarrow V_1$ and $k:U_2\longrightarrow V_2$ such that the following diagram commutes:

$$\begin{array}{ccc} U_1 & \xrightarrow{f'} & V_1 \\ \downarrow h & & \downarrow k \\ U_2 & \xrightarrow{g'} & V_2. \end{array}$$

Since f is a C^∞ stable map germ, we know by Lemma 1.1 that the set germ $S^r(f)_0$ is not empty for $r\leq k$. Let x be any element of $S^r(f')$ with $r<k$. Let $f'_x:(U_1,x)\longrightarrow(V_1,f'(x))$ be a C^∞ map germ determined by f'. Similarly we consider a C^∞ map germ $g'_{h(x)}:(U_2,h(x))\longrightarrow(V_2,g\circ h(x))$. If x is sufficiently near the origin of $\boldsymbol{R}^n$, then both of f'_x and $g'_{h(x)}$ become C^∞ stable map germs. By the construction f'_x and $g'_{h(x)}$ are topologically equivalent. Hence it follows from the assumption of induction that $h(x)$ belongs to $S^r(g')$. This shows that the set germ $S^r(f)_0$ is equivalent to the set germ $S^r(g)_0$ for $r<k$. Hence $S^r(g)_0$ is not empty. If $j<k-1$,

then $S^{k-1}(g)_0$ is empty. Therefore we may suppose that $j=k-1$. By the above argument we know that $S^r(f)_0$ is equivalent to $S^r(g)_0$ for $r<k-1$. Hence $S^{k-1}(f)_0\cup S^k(f)_0$ must be equivalent to $S^{k-1}(g)_0$. Since $S^{k-1}(g)_0$ is a set germ of the differentiable manifold $S^{k-1}(g')$, we have that

$$H_i(S^{k-1}(g)_0\,;\boldsymbol{Z}_2)_{\text{loc}}=\begin{cases}\boldsymbol{Z}_2 & i=\dim\ S^{k-1}(g)\\ 0 & \text{otherwise}\end{cases}$$

(see the definition of $H_i(*\,;\boldsymbol{Z}_2)_{\text{loc}}$ in [1, Section 2]).

Now we consider $f':U\longrightarrow V$ for which $j^1(f'):U\longrightarrow J^1(U,V)$ is transverse to $\Sigma^k(U,V)$ at the origin. When U and V are sufficiently small neighbourhoods, we may identify $J^1(U,V)$ with $U\times V\times\mathrm{Hom}(\boldsymbol{R}^n,\boldsymbol{R}^p)$. Then we can prove that $S^{k-1}(f)_0\cup S^k(f)_0$ is equivalent to the set germ of $\boldsymbol{R}^l\times C(S^{k-1}\times_{Z_2}S^{p-n+k-1})$ at $0\times(*)$ where 0 is the origin, (*) a cone point and, $l=\dim\ S^k(f)_0$. It follows from [1, Lemma 3.2] that $\boldsymbol{R}^l\times C(S^{k-1}\times_{Z_2}S^{p-n+k-1})$ is homeomorphic to $C(S^l(S^{k-1}\times_{Z_2}S^{p-n+k-1}))$ where S^l denotes the l-fold suspension of $S^{k-1}\times S^{p-n+k-1}$. Hence we have that

$$H_{l+2}(S^k(f)_0\cup S^{k-1}(f)_0\,;\boldsymbol{Z}_2)_{\text{loc}}=H_1(S^{k-1}\times_{Z_2}S^{p-n+k-1}\,;\boldsymbol{Z}_2).$$

It is easily seen that

$$H_1(S^{k-1}\times_{Z_2}S^{p-n+k-1}\,;\boldsymbol{Z}_2)\neq\{0\}$$

for $k-1$ and $p-n+k-1\geq 1$. This contradicts to the fact that

$$H_{l+2}(S^{k-1}(g)_0\,;\boldsymbol{Z}_2)_{\text{loc}}=\{0\},$$

since $l+2<\dim\ S^{k-1}(g)_0=(k-1)+(p-n+k-1)+l+1$. This completes the proof.

Proof of Corollary. Since f and g are proper C^∞ stable maps, it follows from [8, Theorem 4.1] that for any point x of N, the germs $f_x:(N,x)\longrightarrow(P,f(x))$ and $g_x:(N,x)\longrightarrow(P,g(x))$ determined by f and g are C^∞ stable map germs respectively. Moreover f_x and g_x are topologically equivalent. Hence it follows from Theorem that $\mathrm{kr}(f_x)$ and $\mathrm{kr}(g_x)$ are equal. Therefore if $h:N\longrightarrow N$ and $k:P\longrightarrow P$ are homeomorphisms such that $g\circ h=k\circ f$, then h maps $S^i(f)$ onto $S^i(g)$. This is what we want.

5. Examples. We consider the special case of Corollary. Let f be a proper submersion of N into P. If g is a proper C^∞ stable map and topologically equivalent to f, then it follows from Corollary that g must be a submersion of N into P. We now construct such examples. Let Σ be an exotic sphere of dimension n (see [5]). By a standard argument in differential topology and the h-cobordism theorem (see [9]) we can prove that $\Sigma \times D^k$ is diffeomorphic to $S^n \times D^k$ for a sufficiently large integer k. By attaching two copies of $\Sigma \times D^k$ and $S^n \times D^k$ on their boundaries respectively we obtain a diffeomorphism $h: \Sigma \times S^k \longrightarrow S^n \times S^k$. Now we consider two projections $p_1: S^n \times S^k \longrightarrow S^k$ and $p_2: \Sigma \times S^k \longrightarrow S^k$. Then we have two submersions, p_1, $p_2 \circ h^{-1}: S^n \times S^k \longrightarrow S^k$. Since Σ is homeomorphic to S^n $(n \geq 5)$, we have that p_1 and $p_2 \circ h^{-1}$ are topologically equivalent. However they are not differentiably equivalent since the fibre of $p_2 \circ h^{-1}$ is the exotic sphere Σ.

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