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ON A THEOREM OF M.S. PUTCHA AND A. YAQUB

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Recently, M.S. Putcha and A. Yaqub [3] proved the following : Let S be a multiplicative subsemigroup of the ring $M_n(F)$ of all $n\times n$ matrices over an arbitrary field F. Suppose that S contains all scalar matrices and suppose, further, that $a\in S$ always implies that $a+I\in S$, where I denotes the identity $n\times n$ matrix. Then S is a subalgebra of $M_n(F)$.

Our present objective is to prove the following theorem and deduce several generalizations of the above result.

Theorem. *Let S be a multiplicative subsemigroup of a ring R with 1. Suppose that S is strongly π-regular and suppose, further, that $a\in S$ always implies that $-a\in S$ and $a+1\in S$. Then S is a subring of R.*

In preparation for the proof of our theorem, we establish the following lemmas.

Lemma 1. *Let S be a semigroup, and a a strongly π-regular element of S, namely $a^n=a^{2n}b=ca^{2n}$ for some positive integer n and some $b, c\in S$. Let $d=a^nb^2$ and $e=a^nd$. Then $ad=da$ and e is an idempotent such that $ae=ea$ and $a^ne=a^{2n}d=a^n$.*

Proof. See the proof of [1, Lemma 1].

Lemma 2. *Let S be as in Theorem. Let $a, b\in S$.*
(1) *If $ab=0$ then $a+b\in S$.*
(2) *If a is invertible then $a+b\in S$.*
(3) *If a is nilpotent then $a+b\in S$.*

Proof. (1) $a+b=-\{-(a+1)(b+1)+1\}\in S$.
(2) Since $a^{-1}\in S$ by Lemma 1, we get $a+b=a(1+a^{-1}b)\in S$.
(3) Since $a+1$ is invertible, $a+b=-[-\{(a+1)+b\}+1]\in S$ by (2).

We are now ready to complete the proof of our theorem.

Proof of Theorem. Let a, b be arbitrary elements of S. We have to show that $a+b\in S$. According to Lemma 2 (2) and (3), we may assume that a is neither invertible nor nilpotent. Then, by Lemma 1, we can easily see that S contains a non-trivial idempotent e such that $ae=ea$ is invertible in eRe and $a(1-e)$ is nilpotent. Note that all the hypotheses in

Theorem are inherited by $eSe(\subseteq eRe)$ and $(1-e)S(1-e)(\subseteq(1-e)R(1-e))$. Hence, by Lemma 2 (2) and (3), $e(a+b)e = ae+ebe \in eSe \subseteq S$ and $(1-e)(a+b)(1-e) = a(1-e)+(1-e)b(1-b) \in (1-e)S(1-e) \subseteq S$. Since $e(a+b)e\cdot(1-e)(a+b)(1-e) = 0$ and both $e(a+b)(1-e) = eb(1-e)$ and $(1-e)(a+b)e = (1-e)be$ are nilpotent elements in S, Lemma 2 (1) and (3) prove that $a+b = e(a+b)e+(1-e)(a+b)(1-e)+e(a+b)(1-e)+(1-e)(a+b)e \in S$.

In advance of stating the first corollary, we introduce the following definition: A ring A with 1 is said to be *right integral* over a unital subring B, if for each $a \in A$ there exists a positive integer n such that $\sum_{i=0}^{\infty} a^iB = \sum_{i=0}^{n} a^iB$.

Corollary 1. *Let R be a right integral extension of a division ring D. Let S be a multiplicative subsemigroup of R. Suppose that S contains D and suppose, further, that $a \in S$ always implies that $a+1 \in S$. Then S is a subring of R.*

Proof. Let a be an arbitrary element of R. Since R is a right integral extension of D, we can easily see that $a^m = a^{m+1}a_0$ with some positive integer m and some $a_0 \in \sum_{i=0}^{\infty} a^iD$. Hence, by [2, Proposition 2], R is strongly π-regular. Henceforth, we let a be an arbitrary element of S. Since every element of $\sum_{i=0}^{\infty} a^iD$ is of the form $a^k(a^ha_h+ \cdots +1)\alpha$ $(\alpha,\ a_i \in D)$, an easy induction proves that $\sum_{i=0}^{\infty} a^iD \subseteq S$. Thus, $a^n = a^{2n}b = ca^{2n}$ for some positive integer n and some $b \in S$ and $c \in R$. Now, by Lemma 1, setting $d = a^nb^2 \in S$, we get $a^{2n}d = da^{2n} = a^n$. This implies that S is a strongly π-regular semigroup. Hence, S is a subring of R by Theorem.

The following are immediate consequences of Corollary 1.

Corollary 2. *Let S be a multiplicative subsemigroup of an algebraic algebra R with 1 over a field F. Suppose that S contains $F(=F\cdot 1)$ and suppose, further, that $a \in S$ always implies $a+1 \in S$. Then S is a subalgebra of R.*

Corollary 3. *Let D be a division ring, and S a multiplicative subsemigroup of $M_n(D)$. Suppose that S contains all scalar matrices and suppose, further, that $a \in S$ always implies that $a+I \in S$. Then S is a subring of $M_n(D)$.*

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