Math. J. Okayama Univ. 24 (1982), 15—19

SOME REMARKS ON BISIMPLE RINGS

YASUYUKI HIRANO and HISAO TOMINAGA

Throughout, R will represent an (associative) ring, and $Z_r(R)$ (resp. $Z_l(R)$) the right (resp. left) singular ideal of R. A ring R ($\neq 0$) is called an *s-unital* ring if for each $x \in R$ there holds $x \in Rx \cap xR$. As stated in [4], if R is an s-unital ring, then for any finite subset F of R there exists an element e in R such that $ex=xe=x$ for all $x \in F$. Consequently, for any finite subset F of a regular ring R there exists an idempotent e in R such that $F \subseteq eRe$. A ring $R(\neq 0)$ is called *pseudobisimple* if for each pair of non-zero elements a, b in R there exists an element c in R such that $aR = cR$ and $Rb = Rc$. Following [5], an s-unital pseudobisimple ring is called a *bisimple* ring.

Obviously, every zero-ring is a pseudobisimple ring, $R = \begin{bmatrix} 0 & \boldsymbol{Z} & \boldsymbol{Z} \\ 0 & 0 & \boldsymbol{Z} \\ 0 & 0 & 0 \end{bmatrix}$ is a pseudobisimple ring with $R^2 \neq 0$ and $R^3 = 0$, and every division ring is a bisimple ring. Now, let α and β be infinite cardinals with $\alpha \leq \beta$, and V a left vector space of dimension β over a division ring D. Write $S = \{a \in \mathrm{End}_D V \mid \mathrm{rank}\ \ a \leq \alpha\}$ and $T = \{a \in \mathrm{End}_D V \mid \mathrm{rank}\ \ a < \alpha\}$. Then $B(D;\alpha,\beta) = S/T$ is a bisimple ring (see [5, (1.1)]).

The purpose of this note is to prove the following theorems which improve Lemma 1.3, Theorem 1.4, Corollary 1.5 and Corollary 2.3 of [5].

Theorem 1. *The following are equivalent:*

1) *R is a regular bisimple ring.*

2) *R is a bisimple ring containing a non-zero idempotent.*

3) *R is a non-zero regular ring whose non-zero principal right (resp. left) ideals are isomorphic as right (resp. left) R-modules.*

Theorem 2. *Let R be a bisimple ring. Then the following are equivalent:*

1) *The set E of idempotents in R is a non-zero multiplicative semigroup.*

2) *R is of bounded (nilpotency) index.*

3) *R satisfies the minimal condition on principal left ideals.*

4) *R satisfies the maximal condition on principal right ideals.*

5) *R satisfies the minimal condition on left annihilators.*

6) *R satisfies the maximal condition on right annihilators.*
7) *R has finite Goldie dimension.*
8) *R is a division ring.*
3)′—7)′: *The left-right analogues of* 3)—7).

In preparation for the proofs of our theorems, we state two lemmas.

Lemma 1. *Let R be an s-unital ring.*

(1) *If R is simple, then R is fully left and right idempotent and $Z_r(R) = Z_l(R) = 0$.*

(2) *If R contains a left (or right) identity then R has* 1.

(3) *If R satisfies the maximal condition on left (or right) annihilators then R has* 1.

(4) *Let a be an element of R. Then Ra (resp. aR) is a direct summand of ${}_RR$ (resp. R_R) if and only if a is (von Neumann) regular.*

(5) *Let a be an element of R. If Ra (resp. aR) is maximal among the principal left (resp. right) ideals then a is regular.*

Proof. (1) For any non-zero $a \in R$ we have $RaR = R$. So, $(Ra)^2 = Ra$ and $(aR)^2 = aR$. Now, $Z_r(R) = Z_l(R) = 0$ by [6, Proposition 7 (1)].

(2) Let e be a left identity of R. Then for any $x \in R$ we have $x-xe \in (x-xe)R = (x-xe)eR = 0$, namely e is the identity of R.

(3) By [3, Theorem 4] and (2).

(4) By [7, Lemma 1 (3)].

(5) Let e be an element in R with $a = ae = ea$. Then we have $Ra = Re$, and so $e = a'a$ for some a'. Thus, $a = aa'a$.

Lemma 2 (cf. [5, Lemmas 1.2 and 1.3]). (1) *Let R be a pseudo-bisimple ring. Then either $R^3 = 0$ or R is a subdirectly irreducible ring with heart R^3 (and R^3 is a simple ring). In particular, if R is bisimple then R is a simple primitive ring and $Z_r(R) = Z_l(R) = 0$.*

(2) *Let R be a bisimple ring. Then any two non-zero principal right (resp. left) ideals of R are isomorphic as right (resp. left) R-modules.*

(3) *If e is a non-zero idempotent of a pseudobisimple ring R then eRe is a bisimple ring with identity.*

(4) *If e and f are non-zero idempotents in a pseudobisimple ring R then eRe is isomorphic to fRf.*

Proof. Let a, b be arbitrary non-zero elements in R, and let c be such that $aR = cR$ and $Rb = Rc$.

(1) Since $RaR = RcR = RbR$, the first assertion is easily seen. Henceforth, we assume that R is bisimple. Then R is simple, and so $Z_r(R) = Z_l(R) = 0$ by Lemma 1 (1). Now, suppose that R is quasi-regular, and choose an element e with $a = ae$. Then we have a contradiction $a = a(1-e)(1-e)^{-1} = 0$, and hence R must be primitive.

(2) Choose $u, v \in R$ such that $b = uc$ and $c = vb$. Then the R-homomorphisms $\phi : aR \to bR$ defined by $cx \mapsto ucx$ and $\psi : bR \to aR$ defined by $bx \mapsto vbx$ are mutually converse.

(3) and (4) If a, b are in eRe then $c = ece$, and therefore $a(eRe) = c(eRe)$ and $(eRe)b = (eRe)c$. (4) is obvious by Lemma 2 (2).

Corollary 1. *Let D be a division ring, and α, β infinite cardinals with $\alpha \leq \beta$. If e is a non-zero idempotent in $B(D;\alpha,\beta)$ then $eB(D;\alpha,\beta)e \simeq B(D;\alpha,\alpha)$.*

Proof. Let f be an idempotent in $\mathrm{End}_D V$ with rank $f = \alpha$. Then it is easy to see that $\bar{f}B(D;\alpha,\beta)\bar{f} \simeq B(D;\alpha,\alpha)$. On the other hand, by Lemma 2 (4), $\bar{f}B(D;\alpha,\beta)\bar{f} \simeq eB(D;\alpha,\beta)e$, and therefore we obtain the assertion.

Corollary 2. *Let R be a regular bisimple ring. If R is not a division ring, then for any finite subset F of R there exists a non-zero idempotent e in R such that eRe contains F and $(eRe)_n \simeq eRe$ for any positive integer n; in particular, $(R)_n$ is a regular bisimple ring for any positive integer n.*

Proof. Let e be a non-zero idempotent in R such that $F \subseteq eRe$. According to Lemma 2 (3), eRe is a regular bisimple ring with identity. Thus, henceforth, we may assume that R has 1. Since R is not a division ring but a regular ring, by making use of Lemma 2 (2) we can easily see that R is isomorphic to the direct sum of n copies of R as right R-module. Hence $(R)_n$ is isomorphic to the bisimple ring R.

We are now ready to complete the proofs of our theorems.

Proof of Theorem 1. 1) $\Rightarrow$ 3). By Lemma 2 (2).

2) $\Rightarrow$ 1). Let e be a non-zero idempotent, and a an arbitrary non-zero element of R. Then there exists $c \in R$ such that $eR = cR$ and $Ra = Rc$. Now, by Lemma 1 (4), a is regular.

3) $\Rightarrow$ 2). Let a, b be arbitrary non-zero elements in R, and $Rb = Re$ with an idempotent e. By hypothesis, there exists an R-isomorphism

$\phi: eR \to aR$. We let $c = \phi(e)$, and $Rc = Rf$ with an idempotent f. Clearly, $cR = aR$ and $r_R(e) = r_R(c) = r_R(f)$. Now, let g be an idempotent in R such that $e, f \in gRg$. Since $(1-f)g$ is in $r_R(f)$, we get $0 = e(1-f)g = e - ef$, namely $e = ef$. Similarly, we have $f = fe$. Hence, $Rb = Re = Rf = Rc$.

A ring R is called *strictly prime* if for each non-zero $a \in R$ there holds $r_R(ab) = 0$ with some $b \in R$. The next improves [2, Theorem 2.1].

Corollary 3. *Let R be a ring with 1. If R is right self-injective, then the following are equivalent*:

1) *R is bisimple.*
2) *R is strictly prime.*
3) *Every non-zero principal right ideal of R is isomorphic to R as right R-module.*
4) *Either R is a division ring, or else R is a directly infinite simple ring.*
5) *Either R is a division ring, or else R is a simple ring and $R_R \simeq (R \oplus R)_R$.*

Proof. 5) $\Leftrightarrow$ 4) $\Leftrightarrow$ 2) $\Rightarrow$ 3). According to [2, Theorem 2.1], it suffices to show that if 2) is satisfied then R is simple. First, we prove that R is right non-singular. Suppose that $r_R(a)$ is essential in R_R for some $a \in R$. If a is non-zero, then $r_R(ab) = 0$ for some $b \in R$. Since $r_R(a) \cap bR$ is non-zero, we have a contradiction $r_R(ab) \neq 0$. Hence, R coincides with its maximal right quotient ring that is a regular ring. Now, let c be an arbitrary non-zero element of R, and d such that $r_R(cd) = 0$. Since $cdR_R \simeq R_R$ and R is a regular ring, it is easy to see that $R = RcdR \subseteq RcR$, namely R is simple.

1) $\Rightarrow$ 2). Let a be an arbitrary non-zero element in R, and c such that $aR = cR$ and $Rc = R$. Choose an element b in R such that $ab = c$. Then $r_R(ab) = r_R(c) = 0$.

3) $\Rightarrow$ 1). Since $aR_R(\simeq R_R)$ is injective for any non-zero $a \in R$, R is a regular ring. Hence, R is bisimple by Theorem 1.

Proof of Theorem 2. Obviously, 8) implies 1)—7), and 5) is equivalent to 6).

1) $\Rightarrow$ 8). According to Theorem 1, R is a regular ring. If R is not a division ring, E cannot form a semigroup by Corollary 2. Hence R is a division ring.

2) $\Rightarrow$ 3). Let a be an arbitrary non-zero element of R, and e such that $ae = ea = a$. Then, there exists a non-zero element c such that $Re = Rc$ and $aR = cR$, and so $e = xc$ and $c = ay$ for some $x, y \in R$. Since $xc^2 = c$, c is strongly regular by [1, Theorem 1]. Hence, R is regular by Theorem 1. If R is not a division ring, then R cannot be of bounded index by Corollary 2.

3) $\Rightarrow$ 8). By Lemma 2 (1), R is a direct sum of minimal left ideals. Hence, by Lemma 1 (4) and Theorem 1, R is a regular ring. Combining this with Lemma 2 (2), we see that R itself is a minimal left ideal.

4) $\Rightarrow$ 6). By Lemma 1 (5) and Theorem 1, R is a regular ring. Hence, there holds 6).

6) $\Rightarrow$ 8). R is a regular ring with 1 by Lemma 1 (3) and Theorem 1. Now, by Lemma 2 (2), we can easily see that R_R is irreducible. Thus R is a division ring.

7) $\Rightarrow$ 6). Since $Z_r(R) = 0$ by Lemma 2 (1), [8, Lemma 3] enables us to see that R satisfies the maximal condition on right annihilators.

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DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY

(Received December 16, 1981)