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SOME POLYNOMIAL IDENTITIES AND COMMUTATIVITY OF s-UNITAL RINGS

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Throughout this paper, R will represent an (associative) ring (with or without identity 1), $C = C(R)$ the center of R, $D = D(R)$ the commutator ideal of R, and $N = N(R)$ the set of all nilpotent elements in R.

A ring R is called *s-unital* if $x \in Rx \cap xR$ for any $x \in R$. As stated in [13], if R is s-unital, then for any finite subset F of R there exists an element e in R such that $ex = xe = x$ for all $x \in F$. Such an element e will be called a *pseudo-identity* of F (in R).

Let n be a positive integer. We consider the following ring-properties:

$P_1(n)$: $(xy)^n = x^n y^n$ and $(xy)^{n+1} = x^{n+1}y^{n+1}$ for all $x, y \in R$.
$P_2(n)$: $(xy)^n = x^n y^n = y^n x^n$ for all $x, y \in R$.
$P_3(n)$: $(xy)^n = (yx)^n$ for all $x, y \in R$.
$P_4(n)$: $[x, (xy)^n] = 0$ for all $x, y \in R$.
$P_5(n)$: $[x, (yx)^n] = 0$ for all $x, y \in R$.
$P_6(n)$: $[x, y^n] = 0$ for all $x, y \in R$.
$P_7(n)$: $[x, y^n] = [x^n, y]$ for all $x, y \in R$.
$P_8(n)$: There is a polynomial $\psi(\lambda)$ with integer coefficients such that $[x, y^n] = [\psi(x), y]$ for all $x, y \in R$.
$P_9(n)$: $[x, (x+y)^n - y^n] = 0$ for all $x, y \in R$.
$P_{10}(n)$: $[x^n, y^n] = 0$ for all $x, y \in R$.
$Q(n)$: For any $x, y \in R$, $n[x, y] = 0$ implies $[x, y] = 0$.

The properties $P_1(n)$, $P_3(n)$, $P_6(n)$, $P_7(n)$ and $P_{10}(n)$ have been considered by many authors. The main objective of this paper is to prove the following

Theorem 1. *Let $m_1, \cdots, m_t$ and $n_1, \cdots, n_t$ be (fixed) positive integers such that $1 \leq m_i \leq 9$ and $2 \leq n_i$ for $i = 1, \cdots, t$. Let $d = (n_1, \cdots, n_t)$. If an s-unital ring R has the (conjunctive) property $P_{m_1}(n_1) \wedge \cdots \wedge P_{m_t}(n_t) \wedge Q(d)$, then R is commutative.*

In preparation for the proof of our theorem, we introduce here some definitions. Let P be a ring-property. If P is inherited by every subring and every homomorphic image, P is called an *h-property*. More weakly, if P is inherited by every finitely generated subring and every natural homomorphic image modulo the annihilator of a central element, P is called an *H-property*. And, a ring-property P such that a ring has the property P if and only if all its finitely generated subrings have P, is called an *F-property*. Finally, P is called a $C(n)$-property if every ring with 1 having the property $P \wedge Q(n)$ is commutative.

Obviously, $P_1(n) - P_{10}(n)$ are h-properties and $Q(n)$ is an H-property. These properties are also F-properties and the property "being commutative" is an F-property.

To our end, we shall prove three propositions. The first one enables us to reduce some problems of s-unital rings into those of rings with 1.

Proposition 1. *Let P be an H-property, and P' an F-property. If every ring with 1 having the property P has the property P', then every s-unital ring having P has P'.*

Proof. Let R be an s-unital ring having the property P. We show that if F is a finite subset of R, then the subring $\langle F \rangle$ generated by F has the property P'. To see this, choose a pseudo-identity e of F and a pseudo-identity e' of $F \cup \{e\}$. Obviously, e is a central element of $S = \langle F \cup \{e, e'\} \rangle$. Let A be the annihilator of e in S. Then the factor ring S/A has the identity $e' + A$. Since $\langle F \rangle \cap A = 0$, $\langle F \rangle$ may be regarded as a subring of S/A. Thus, by hypothesis, $\langle F \rangle$ has the property P'.

Some known results on rings with 1 can be extended to s-unital rings by Proposition 1. For example, by [11, Theorem 3] and [4, Theorem 1] we obtain

Corollary 1. *Let R be an s-unital ring.*

(1) *Let k be a positive integer. Suppose that for each pair of elements x, y in R there exist positive integers m, n such that $_k[x^m, y^n] = 0$. Then D is a nil ideal.*

(2) *Suppose that for each pair of elements x, y in R there exists an integer $n \geq 2$ such that $(xy)^n = x^n y^n$ and $(xy)^{n+1} = x^{n+1} y^{n+1}$. Then D is a nil ideal.*

Next, we reprove a theorem of Kezlan [10].

Proposition 2. *Let f be a polynomial in non-commuting indeterminates $x_1, \cdots, x_k$ with integer coefficients. Then the following statements are equivalent:*

1) *For any ring R satisfying $f = 0$, D is a nil ideal.*
2) *Every semiprime ring satisfying $f = 0$ is commutative.*
3) *For every prime p, $(GF(p))_2$ fails to satisfy $f = 0$.*

Proof. Since 2) ⇒ 1) ⇒ 3) are immediate, it remains only to prove that 3) implies 2). Obviously, the coefficients of f are relatively prime. Since every semiprime ring is a subdirect sum of prime rings, it suffices to show that every prime ring R satisfying $f = 0$ is commutative. Now, by a theorem of Amitsur [1, Theorem 7 (6)], the (classical) quotient ring R^* of R is an Artinian simple ring satisfying $f = 0$. Hence, by 3) (and Posner's theorem), R^* is a central division algebra of finite rank m^2 over $C^* = C(R^*)$. Suppose that R^* is not commutative, namely $m \geq 2$, and choose a maximal subfield K of R^*. Then again by the theorem of Amitsur, $R^* \otimes_{C^*} K \simeq (K)_m$ satisfies $f = 0$. But this cotradicts 3). Thus, R^*, and therefore R is commutative.

Corollary 2 (cf. [5, Theorems 1, 2, 3] and [9, Theorem]). *Let R be a semiprime ring, and ν a (fixed) positive integer.*

(1) *If for each pair of elements x, y in R there exists an integer n such that $2 \leq n \leq \nu$ and $[x, (xy)^n - x^n y^n] = 0$ (resp. $[x, (xy)^n - (yx)^n] = 0$), then R is commutative.*

(2) *Suppose that for each pair of elements x, y in R there exists an integer n such that $2 \leq n \leq \nu$ and $[x, [x^n, y] - [x, y^n]] = 0$. Then R is commutative.*

Proof. (1) In fact, R satisfies the identity

$$f(x,y,z) = [x,(xy)^2 - x^2y^2]z[x,(xy)^3 - x^3y^3]z \cdots [x,(xy)^\nu - x^\nu y^\nu] = 0$$

(resp. $f(x,y,z) = [x,(xy)^2 - (yx)^2]z[x,(xy)^3 - (yx)^3]z \cdots [x,(xy)^\nu - (yx)^\nu] = 0$), but $f(E_{12}, E_{21}, E_{21}) \neq 0$ in $(GF(p))_2$.

(2) R satisfies the identity

$$f(x,y,z) = [x,[x^2,y] - [x,y^2]]z[x,[x^3,y] - [x,y^3]]z \cdots [x,[x^\nu,y] - [x,y^\nu]] = 0,$$

but $f(E_{11}, E_{12}, E_{21}) \neq 0$ in $(GF(p))_2$.

According to Proposition 2, as Corollary 2 shows, various kinds of semiprime PI-rings (especially, semiprime rings having any one of the properties $P_1(n) - P_{10}(n)$ $(n \geq 2)$) are commutative. However, if we remove the hypothesis "semiprime", even under some extra hypothesis, say that R has 1 or that R is n-torsion free, we have not yet obtained definite results concerning the precise commutativity of R.

In the subsequent study, we shall use freely the following well-known results: Let a, $b \in R$, and n a positive integer.

(I) If $[a,[a,b]] = 0$ then $[a^n,b] = na^{n-1}[a,b]$.

(II) If R contains 1 and $a^n b = (a+1)^n b = 0$, then $b = 0$.

Now, in advance of exposing the relationship among the properties $P_1(n) - P_{10}(n)$, we state the following lemma.

Lemma 1. *Let* $n \geq 2$. *If* $[x,y] \in C$ *for all* x, $y \in R$, *then* $P_7(n)$ *implies* $P_6(n^4)$.

Proof. We claim that $[x,y^{n^2}]x^{(n-1)^2} = [x,y^{n^2}]$ for all x, $y \in R$. Indeed, by (I) we have $[x,y^{n^2}]x^{(n-1)^2} = x^{(n-1)^2}[x^n,y^n] = nx^{n(n-1)}[x,y^n] = nx^{n(n-1)}[x^n,y] = [x^{n^2},y] = [x,y^{n^2}]$. Now, by making use of the argument employed in the proof of [7, Lemma 5], we can prove that the subring $\langle x^{n^2} \mid x \in R \rangle$ is commutative. This implies that $[x^{n^4},y] = [x^{n^2},y^{n^2}] = 0$ for all x, $y \in R$.

Proposition 3. (i) *If R is s-unital, then* $P_1(n) \Leftrightarrow P_2(n) \Rightarrow P_3(n) \Rightarrow P_4(n) \Leftrightarrow P_5(n) \Leftrightarrow P_6(n) \Rightarrow P_{10}(n)$ *and* $P_6(n) \Rightarrow P_7(n)$.

(ii) *If* $n \geq 2$, *then* $P_7(n) \Leftrightarrow P_8(n) \Leftrightarrow P_9(n) \Rightarrow P_6(n^a)$ *for some positive integer* a.

Proof. (i) In view of Proposition 1, we may assume that R has 1. Obviously, $P_3(n) \Rightarrow P_4(n) \wedge P_5(n)$, $P_2(n) \Rightarrow P_1(n) \wedge P_3(n)$, and $P_6(n) \Rightarrow P_4(n) \wedge P_5(n) \wedge P_7(n) \wedge P_{10}(n)$. Furthermore, $P_1(n)$ together with $P_6(n)$ implies $P_2(n)$, and so we prove that $P_1(n) \Rightarrow P_4(n)$ (resp. $P_5(n)) \Rightarrow P_6(n)$.

$P_1(n) \Rightarrow P_4(n)$. Since $xy \cdot x^n y^n = (xy)^{n+1} = x^{n+1}y^{n+1}$, we get $x[x^n,y]y^n = 0$. Hence $x[x^n,y] = 0$ by (II), and in particular $x[x^n,y^n] = 0$. So we have

$$[x,(xy)^n] = x\{(xy)^n - (yx)^n\} = x[x^n,y^n] = 0.$$

$P_4(n) \Rightarrow P_6(n)$. By [12, Theorem], there exists a positive integer k such that $kD = 0$. If u is in N, then for any $x \in R$ we have

$$[x^n,u] = [\{(1+u)(1+u)^{-1}x\}^n, 1+u] = 0.$$

Hence, noting that $D \subseteq N$ by Proposition 2, we see that $[x^n,[x^n,y]] = 0$, and then $[x^{nk},y] = kx^{n(k-1)}[x^n,y] = 0$ by (I). This enables us to see that $x^{n^2k}[x,y^n] = [x,x^{n^2k}y^n] = [x,(x \cdot x^{nk-1}y)^n] = 0$. Thus, $[x,y^n] = 0$ by (II).

Similarly, we can prove that $P_1(n) \Rightarrow P_5(n) \Rightarrow P_6(n)$.

(ii) Obviously, $P_7(n)$ implies $P_8(n)$. If R has $P_8(n)$, then

$$[x,(x+y)^n - y^n] = [\psi(x),(x+y) - y] = 0.$$

Next, if R has $P_9(n)$, then

$$[x,y^n]-[x^n,y]=[x,(x+y)^n]-[(x+y)^n,y]=[x+y,(x+y)^n]=0.$$

We have thus seen the equivalence of $P_7(n)-P_9(n)$.

Now, suppose R has the property $P_7(n)$. By [7, Lemma 1] there exists a positive integer h such that $[x,y]^h=0$ for all $x,\ y\in R$. Choose a positive integer k such that $n^k\geq h$, and let $T=\langle x^{n^k}\mid x\in R\rangle$. Since $[[x,y],z^{n^k}]=[[x,y]^{n^k},z]=0$ for all $x,\ y,\ z\in R$, Lemma 1 shows that $[s^{n^4},t]=0$ for all $s,\ t\in T$. It therefore follows that $[x^{n^{2k+4}},y]=[x^{n^{k+4}},y^{n^k}]=0$ for all $x,\ y\in R$.

Remark 1. Let $i,\ j$ be non-negative integers. Let us consider the following ring-property:

$P(i,j;n)$: $[x,(x^iyx^j)^n]=0$ for all $x,\ y\in R$.

Obviously, $P(1,0;n)=P_4(n)$, $P(0,1;n)=P_5(n)$ and $P(0,0;n)=P_6(n)$. From the proof of Proposition 3 (i), we can easily see that $P(i,j;n)$ is equivalent to $P_6(n)$ for any $i,\ j\geq 0$.

Obviously, if the power map π_n: $x\mapsto x^n$ is a ring-homomorphism of R then R has the property $P_9(n)$. In [6, Theorem 3], it is shown that if π_n is a surjective ring-homomorphism of R for some $n\geq 2$ then R is commutative. On the other hand, in [3, Theorem 3], it is shown that if a ring R with 1 has the property $P_1(n)$ and is generated by $\{x^{n^2}\mid x\in R\}$ or $\{x^{n(n+1)}\mid x\in R\}$ then R is commutative. The next improves these results as well as [2, Corollary 2] (see also [7, Corollary 2]).

Corollary 3. *Let $n\geq 2$. Let R be an s-unital ring having one of the properties $P_1(n)-P_6(n)$ or a ring having one of the properties $P_7(n)-P_9(n)$, and let $T=\langle x^n\mid x\in R\rangle$. If the centralizer of T in R coincides with C, then R is commutative.*

Proof. If an s-unital ring R has one of the properties $P_1(n)-P_6(n)$, then it has the property $P_6(n)$. So the assertion is clear. If a ring R has one of the properties $P_7(n)-P_9(n)$, then it has the property $P_6(n^a)$ for some positive a. Hence $[x^{n^{a-1}},y^n]=[x^{n^a},y]=0$ for all $x,\ y\in R$. Then, $[x^{n^{a-1}},y]=0$ by hypothesis. We can thus continue the same procedure to obtain the conclusion $[x,y]=0$.

Corollary 4. *If $n\geq 2$, then the properties $P_1(n)-P_9(n)$ are $C(n)$-properties.*

Proof. Let R be a ring with 1 having the property $P_i(n) \wedge Q(n)$. If $1 \leq i \leq 6$ then, according to Proposition 3 (i), we may assume that $i = 6$. Given $u \in N$, by an easy induction on the nilpotency index of u, we can show that $u \in C$, and therefore $D \subseteq C$ by Proposition 2. Now, by (I), for any $x, y \in R$ we have $nx^{n-1}[x,y] = [x^n,y] = 0$, whence $[x,y] = 0$ follows by (II). On the other hand, if $7 \leq i \leq 9$, then R has $P_6(n^a)$ and $Q(n^a)(= Q(n))$ for some positive integer a (Proposition 3 (ii)). Hence, R is commutative by what was just proved above.

Proof of Theorem 1. In virtue of Proposition 1, we may assume that R has 1. Since $P_1(n) - P_9(n)$ are $C(n)$-properties, the proof of our theorem is now immediate by [8, Proposition 1].

Corollary 5. *Let* $m_1, \cdots, m_t$ *and* $n_1, \cdots, n_t$ *be (fixed) positive integers such that* $1 \leq m_i \leq 9$, $2 \leq n_i$ $(i = 1, \cdots, t)$ *and* $(n_1, \cdots, n_t) = 1$. *If an s-unital ring* R *has the property* $P_{m_1}(n_1) \wedge \cdots \wedge P_{m_t}(n_t)$, *then* R *is commutative.*

Remark 2. Let $n \geq 2$, and m a strictly proper divisor of n. Then, the properties $P_3(n) - P_{10}(n)$ are not $C(m)$-properties. In this sense, the results on $P_3(n) - P_9(n)$ in Corollary 4 are best possible. To see this, we take a prime divisor p of n such that $p \nmid m$. Let S be a non-commutative algebra over $\mathrm{GF}(p)$ such that $S^3 = 0$. Let R be the ring whose additive group is the direct sum of $\mathrm{GF}(p)$ and S with multiplication given by $(k,s)(k',s') = (kk',ks'+k's+ss')$. It is easy to see that R has the properties $P_3(n) - P_9(n)$ and $Q(n)$, but R is not commutative. In the same way, the properties $P_1(n)$ and $P_2(n)$ are not $C(m)$-properties when n is odd. However, as is easily seen, $P_1(2)$ and $P_2(2)$ are $C(1)$-properties. In general, when n is even, we can not deny the possibility that $P_1(n)$ and $P_2(n)$ can be $C(n/2)$-properties (see [12, Examples 3 and 4]).

Remark 3. So far we did say little about $P_{10}(n)$. It is easy to see that $P_{10}(2)$ is a $C(2)$-property. However, $P_{10}(n)$ is not a $C(n)$-property if n has a divisor of the form $1+p^r+p^{2r}+\cdots+p^{sr}$, where r and s are positive integers and p is a prime not dividing n. In fact, let n have such a divisor and let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a^{p^r} \end{pmatrix} \middle| \; a, b \in \mathrm{GF}(p^{r(s+1)}) \right\}.$$

Then, R is an n-torsion free ring with 1 and has the property $P_{10}(n)$, but

R is not commutative. Thus, in particular, $P_{10}(n)$ is not a $C(n)$-property if $3 \leq n \leq 10$. What about $P_{10}(11)$?

Remark 4. In view of Remark 3, it seems unavoidable to exclude the property $P_{10}(n)$ from the statment in Corollary 5. However, we have the following: If an s-unital ring R has the property $P_i(m) \wedge P_j(n)$ for some positive integers i, j, m and n such that $1 \leq i, j \leq 10$, $2 \leq m, n$ and $(m, n) = 1$, then R is commutative. In fact, if R has this property, then R has the property $P_{10}(m^a) \wedge P_{10}(n^a)$ for some positive integer a (Proposition 3 (ii)). Then R is commutative by Proposition 1 and [14, Theorem] (cf. the proof of [8, Theorem 1]).

REFERENCES

[1] S.A. AMITSUR: Prime rings having polynomial identities with arbitrary coefficients, Proc. London Math. Soc. (3), 17 (1967), 470—486.

[2] H.E. BELL: On some commutativity theorems of Herstein, Archiv Math. 24 (1973), 34—38.

[3] H.E. BELL: On the power map and ring commutativity, Canad. Math. Bull. 21 (1978), 399—404.

[4] B. FELZENSWALB: On the commutativity of rings, Acta Math. Acad. Sci. Hungar. 34 (1979), 257—260.

[5] V. GUPTA: Some remarks on the commutativity of rings, Acta Math. Acad. Sci. Hungar. 36 (1980), 233—236.

[6] I.N. HERSTEIN: Power maps in rings, Michigan Math. J. 8 (1961), 29—32.

[7] Y. HIRANO and H. TOMINAGA: Some commutativity theorems for rings, Hiroshima Math. J. 11 (1981), 457—464.

[8] Y. HIRANO, M. HONGAN and H. TOMINAGA: Supplements to the previous paper "Some commutativity theorems for rings", Math. J. Okayama Univ. 23 (1981), 137—139.

[9] A. KAYA: A theorem on the commutativity of rings, METU J. Pure Appl. Sci. 10 (1977), 261—265.

[10] T.P. KEZLAN: A note on commutativity of semi-prime PI-rings, Math. Japonica 27 (1982), 267—268.

[11] A.A. KLEIN, I. NADA and H.E. BELL: Some commutativity results for rings, Bull. Austral. Math. Soc. 22 (1980), 285—289.

[12] Y. KOBAYASHI: A note on commutativity of rings, Math. J. Okayama Univ. 23 (1981), 141—145.

[13] I. MOGAMI and M. HONGAN: Note on commutativity of rings, Math. J. Okayama Univ. 20 (1978), 21—24.

[14] W.K. NICHOLSON and A. YAQUB: A commutativity theorem for rings and groups, Canad. Math. Bull. 22 (1979), 419—423.

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