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NOTE ON GROUPS WITH ISOMORPHIC GROUP ALGEBRAS

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Introduction. For G a group and R a ring with identity, we denote by RG the group ring of G over R and by $D_{n,R}(G)$ the n-th dimension subgroup modulo R of G. That is, $D_{n,R}(G)$ is defined to be $\{g \in G \mid g-1 \in \Delta_R(G)^n\}$, where $\Delta_R(G)$ is the augmentation ideal of RG. Also, for any normal subgroup N of G we denote by $\Delta_R(G,N)$ the kernel of the natural homomorphism from RG to $R(G/N)$. Note that $\Delta_R(G,N) = \Delta_R(N)RG = RG\Delta_R(N)$.

In this note we prove that if G and H are two groups with isomorphic group algebras over R, where R is an integral domain of characteristic 0 in which no rational prime is invertible, then $D_{n,R}(G) = \{1\}$ if and only if $D_{n,R}(H) = \{1\}$ (Proposition 2.4). The corresponding result for the case where R is the field of p elements for a prime p has been shown by I. B. S. Passi and S. K. Sehgal [3, Corollary 5]. Proposition 2.4 can be combined with [4, 2.4 Corollary] to show that finitely generated nilpotent groups of class 2 are determined by their integral group rings. This result for the finite case is well known (see e.g. [5]).

In the process of establishing Proposition 2.4 we consider the group $V(RG)$ of normalized units of a group ring RG and obtain the following : If G is any group and R is an integral domain of characteristic 0 in which no rational prime is invertible, then for each $n \geqq 1$ the factor group

$$V(RG) \cap (1+\Delta_R(G)^n)/V(RG) \cap (1+\Delta_R(G,D_{n,R}(G)))$$

is torsion free. This is an immediate consequence of Proposition 1.3 which is stated in a more general form.

1. The group of normalized units.

Let RG be the group ring of a group G over a commutative ring R with identity. Denote by $V(RG)$ the group of normalized units of RG, that is, $V(RG) = U(RG) \cap (1+\Delta_R(G))$ where $U(RG)$ is the group of units of RG. We need the following crucial result.

Lemma 1.1 ([6, Corollary II.1.4]). *Let G be a polycyclic-by-finite group and R be an integral domain of characteristic 0 such that no* *element* $g \neq 1$ *of* G *has order invertible in* R. *Suppose that* $u = \sum_g u(g)g \in V(RG)$. *If* u *has finite order and* $u(1) \neq 0$, *then* $u = 1$.

Let $T(G)$ denote the set of torsion elements of a group G.

Lemma 1.2. *Let* G *be a nilpotent group and* R *be an integral domain of characteristic* 0 *in which no element* $g \neq 1$ *of* G *has order invertible. Let* N *be a central subgroup of* G. *If* u *is a unit of* RG *of finite order such that* $u-1 \in \Delta_R(G,N)$, *then* $u = x$ *for some* $x \in N$.

Proof. Since $u-1$ can be written as a finite sum of the form

$$u-1 = \sum_i a_i g_i(x_i-1) \quad (a_i \in R,\ g_i \in G,\ x_i \in N),$$

we may suppose that G is finitely generated and hence $T(G)$ is a finite normal subgroup of G. We proceed by induction on the order of $T(G)$. If $T(G) = \{1\}$, then G is torsion free nilpotent and it follows from [6, Corollary VI.1.7] that $V(RG) = G$. Therefore $u = 1$ and the result is trivial. Assume $T(G) \neq \{1\}$ so that $T(\zeta(G)) \neq \{1\}$, where $\zeta(G)$ is the center of G. Set $W = T(\zeta(G))$ and $\bar{G} = G/W$. Let $\bar{\ }: RG \to R(\bar{G})$ be the natural homomorphism. Then $\bar{u}$ is a unit of $R(\bar{G})$ of finite order such that $\bar{u}-1 \in \Delta_R(\bar{G},\bar{N})$. Since $T(\bar{G}) = T(G)/W$, $T(\bar{G})$ has smaller order than $T(G)$ and $\bar{G}$ has no element $\bar{g} \neq 1$ whose order is invertible in R. Hence by induction, $\bar{u} = \bar{y}$ for some $y \in N$. Since $\bar{y} \in T(\bar{G})$ we see that $y \in W$ and hence $\bar{u} = 1$. Thus $u-1 \in \Delta_R(G,W)$ and there exists an element $x \in W$ with $u(x) \neq 0$. Then $v = x^{-1}u$ is a unit of finite order such that $v \in V(RG)$ and $v(1) \neq 0$. Noting here that G is finitely generated nilpotent and so is polycyclic, it follows by Lemma 1.1 that $v = 1$ and hence $u = x$. Since $u-1 \in \Delta_R(G,N)$ we have $x \in N$ as desired.

Recall that for any (two-sided) ideal I of RG, $G \cap (1+I)$ is a normal subgroup of G. Let $\Delta_R(G) = I_1 \supseteqq I_2 \supseteqq \cdots \supseteqq I_n \supseteqq \cdots$ be a decreasing series of ideals of RG such that $I_1 I_n + I_n I_1 \subseteqq I_{n+1}$ for all $n \geqq 1$. Then $\{G \cap (1+I_n)\}_{n \geqq 1}$ is a descending central series of G, since we have

$$g^{-1}x^{-1}gx-1 = g^{-1}x^{-1}\{(g-1)(x-1)-(x-1)(g-1)\} \in I_{n+1}$$

for $g \in G$ and $x \in G \cap (1+I_n)$. Thus if $G \cap (1+I_n) = \{1\}$ for some n then G is nilpotent. Now, we note that since $\Delta_R(G, G \cap (1+I_n)) \subseteqq I_n$, $V(RG) \cap (1+\Delta_R(G, G \cap (1+I_n)))$ is a normal subgroup of $V(RG) \cap (1+I_n)$.

Proposition 1.3. *Let* G *be an arbitrary group and* R *be an* *integral domain of characteristic* 0 *in which no rational prime is invertible. Let* $\Delta_R(G) = I_1 \supseteqq I_2 \supseteqq \cdots \supseteqq I_n \supseteqq \cdots$ *be a decreasing series of ideals of* RG *such that* $I_1 I_n + I_n I_1 \subseteqq I_{n+1}$ *for all* $n \geqq 1$. *Then for each* $n \geqq 1$ *the factor group*

$$V(RG) \cap (1+I_n)/V(RG) \cap (1+\Delta_R(G,G \cap (1+I_n)))$$

is torsion free.

Proof. We observe that if $\bar{G} = G/G \cap (1+I_n)$, then $\bar{G} \cap (1+\bar{I}_n) = \{1\}$ under the natural homomorphism $\bar{\ } : RG \to R(\bar{G})$. Hence by considering $G/G \cap (1+I_n)$ it suffices to prove the following :

(*) If $G \cap (1+I_n) = \{1\}$, then $V(RG) \cap (1+I_n)$ is torsion free.

We use induction on n to show (*), the case $n = 1$ being trivial. Let $n \geqq 2$ and assume that (*) holds for $n-1$. Set $\bar{G} = G/G \cap (1+I_{n-1})$, and let $\bar{\ } : RG \to R(\bar{G})$ be the natural homomorphism. Then, since $\bar{G} \cap (1+\bar{I}_{n-1}) = \{1\}$, the induction hypothesis shows that $V(R(\bar{G})) \cap (1+\bar{I}_{n-1})$ is torsion free. Let $u \in V(RG) \cap (1+I_n)$ have finite order . Then $\bar{u}$ is a unit of finite order contained in $V(R(\bar{G})) \cap (1+\bar{I}_{n-1})$, so $\bar{u} = 1$. Thus we have

$$u-1 \in \Delta_R(G,G \cap (1+I_{n-1})).$$

Now, since $G \cap (1+I_n) = \{1\}$, $G \cap (1+I_{n-1})$ is a central subgroup of G. It follows from Lemma 1.2 that $u = x$ for some $x \in G \cap (1+I_{n-1})$. However, since $u-1 \in I_n$, $x \in G \cap (1+I_n)$ so $x = 1$. We have therefore seen that $V(RG) \cap (1+I_n)$ is torsion free. This completes the induction on n and (*) is established.

By taking $I_n = \Delta_R(G)^n$ in the above proposition we obtain the following

Corollary 1.4. *If* G *and* R *are as in Proposition* 1.3, *then for each* $n \geqq 1$ *the factor group*

$$V(RG) \cap (1+\Delta_R(G)^n)/V(RG) \cap (1+\Delta_R(G,D_{n,R}(G)))$$

is torsion free.

2. **Isomorphic group algebras.** Let G be a group, R a ring with identity. The following two lemmas are elementary.

Lemma 2.1 (cf. [1, Proposition 1]). *Let* H *be a subgroup of* G.

Then H is finitely generated if and only if $\Delta_R(H)RG$ is finitely generated as a right ideal of RG.

Proof. Assume that H is finitely generated by a finite subset X. Then it is easy to verify that $\Delta_R(H)RG = (X-1)RG$ where $X-1$ is the subset of all elements of the form $x-1$, $x \in X$. Hence $\Delta_R(H)RG$ is a finitely generated right ideal of RG. Conversely, assume that $\Delta_R(H)RG$ is finitely generated by a finite subset S of RG as a right ideal. Then since S is finite there exists a finitely generated subgroup H^* of H such that $S \subseteq \Delta_R(H^*)RG$. Since $S(RG) \subseteq \Delta_R(H^*)RG$ it follows that $\Delta_R(H)RG = \Delta_R(H^*)RG$. Therefore H coincides with H^* and so is finitely generated.

For a subset S of RG, let $l(S)$ denote the left ideal $\{\alpha \in RG \mid \alpha S = 0\}$. Recall that if H is a subgroup of G, then $l(\Delta_R(H)RG) \neq 0$ if and only if H is finite ([1, Proposition 1]).

Lemma 2.2. *Let H be a subgroup of G. Then H is locally finite if and only if for any finite subset S of $\Delta_R(H)RG$ it follows $l(S) \neq 0$.*

Proof. Assume that H is locally finite. Let S be a finite subset of $\Delta_R(H)RG$. Then since H is locally finite there exists a finite subgroup H^* of H such that $S \subseteq \Delta_R(H^*)RG$. As stated above, $l(\Delta_R(H^*)RG) \neq 0$, so it follows $l(S) \neq 0$. Conversely, assume that for any finite subset S of $\Delta_R(H)RG$ it follows $l(S) \neq 0$. Let H^* be a finitely generated subgroup of H. Then by Lemma 2.1 there is a finite subset S of RG with $\Delta_R(H^*)RG = S(RG)$. Since $l(S) \neq 0$, we have $l(\Delta_R(H^*)RG) \neq 0$ so that H^* is finite. Thus we see that H is locally finite.

Now let R be a commutative ring with identity, let G and H be groups having isomorphic group algebras over R. If $\theta : RG \to RH$ is an R-algebra isomorphism then we may suppose that θ is normalized and hence $\theta(\Delta_R(G)) = \Delta_R(H)$ (see [6, p.64]). In what follows, the given R-algebra isomorphism $RG \cong RH$ will be assumed to be normalized.

Lemma 2.3. *Let G be a nilpotent group and R an integral domain. Suppose $RG \cong RH$ as R-algebras. Then $T(H)$ forms a subgroup of H and $G/T(G) \cong H/T(H)$.*

Proof. Note that since G is nilpotent $T(G)$ is a locally finite normal subgroup of G. Let $\theta : RG \to RH$ be an R-algebra isomorphism and set $K = H \cap (1+\theta(\Delta_R(G,T(G))))$. Then since $\Delta_R(H,K) \subseteq \theta(\Delta_R(G,T(G)))$, there is a natural epimorphism $RH/\Delta_R(H,K) \to RH/\theta(\Delta_R(G,T(G)))$, which induces a normalized epimorphism $f: R(H/K) \to R(G/T(G))$ such that $H/K \cap (1+Ker\ f) = \{1\}$. By [6, Corollary VI.1.7] we have $V(R(G/T(G))) = G/T(G)$, so the restriction of f to H/K is an imbedding into $G/T(G)$. This means that f is an isomorphism, and hence we obtain

$$\theta(\Delta_R(G,T(G))) = \Delta_R(H,K) \text{ and } G/T(G) \cong H/K.$$

Consequently, using Lemma 2.2 we deduce that K is locally finite and that H/K is torsion free. Therefore $K=T(H)$ and the proof is complete.

Let $\gamma_n(G)$ denote the n-th term of the lower central series of a group G.

Proposition 2.4. *Let R be an integral domain of characteristic 0 in which no rational prime is invertible. Let G and H be two groups with $RG \cong RH$ as R-algebras. Then*

$$D_{n,R}(G) = \{1\} \textit{ if and only if } D_{n,R}(H) = \{1\}.$$

Proof. Suppose $D_{n,R}(G) = \{1\}$. Then G is nilpotent and thus by Lemma 2.3, $G/T(G) \cong H/T(H)$. Since $\gamma_n(G)=\{1\}$ we have $\gamma_n(G/T(G)) = \{1\}$ and hence $\gamma_n(H/T(H)) = \{1\}$, so that $\gamma_n(H) \subseteq T(H)$. Moreover, we get $D_{n,R}(H) \subseteq T(H)$, because $D_{n,R}(H)/\gamma_n(H)$ is torsion (see e.g. [2, pp.36—37]). Now, by Corollary 1.4, $V(RG) \cap (1+\Delta_R(G)^n)$ is torsion free. Since $\theta(V(RG) \cap (1+\Delta_R(G)^n)) = V(RH) \cap (1+\Delta_R(H)^n)$ under the given R-algebra isomorphism $\theta: RG \to RH$, it follows that $V(RH) \cap (1+\Delta_R(H)^n)$ and hence $D_{n,R}(H)$ is torsion free. However, $D_{n,R}(H)$ is torsion, so we conclude that $D_{n,R}(H) = \{1\}$. The converse follows by symmetry.

Corollary 2.5. *If, under the hypotheses of Proposition 2.4, G is finitely generatea nilpotent, then so is H.*

Proof. Note that with the hypothesis on R, $D_{n,R}(X) = D_{n,Z}(X)$ for any group X where Z is the ring of rational integers (see [2, p.16]). Since G is finitely generated nilpotent we know from [7, Corollary 1] that $D_{n,Z}(G) = \{1\}$ with some n. Thus by Proposition 2.4, $D_{n,R}(H) = \{1\}$, so H is nilpotent. By Lemma 2.1, $\Delta_R(G)$ and therefore $\Delta_R(H)$ is a finitely generated right ideal, so H is finitely generated. Hence the result follows.

It is known [4, 2.4 Corollary] that if G is a finitely generated group, then $ZG \cong ZH$ implies $G/D_{3,Z}(G) \cong H/D_{3,Z}(H)$ (see also [3]). Since $D_{3,Z}(G) = \gamma_3(G)$ for every group G (see e.g. [2, p.66]), we have immediately

Corollary 2.6. *If G is a finitely generated nilpotent group of class 2, then $ZG \cong ZH$ implies $G \cong H$.*

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