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COUNTEREXAMPLES IN THE INDIVIDAL ERGODIC THEORY OF PSEUDO-RESOLVENTS

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1. Introduction. Let D denote the set of all complex numbers λ with $\operatorname{Re}(\lambda)>0$, and D_+ the set of all positive reals. Let $J=(J_\lambda:\lambda\in D)$ be a pseudo-resolvent of bounded linear operators on L_1 of a σ-finite measure space. Thus $J_\lambda-J_\nu=(\nu-\lambda)J_\lambda J_\nu$ for all λ and ν in D. Previously it was proved (cf. [5]) that if J satisfies that

$$\|\lambda J_\lambda\|_1\le 1 \text{ for all } \lambda\in D_+$$

and also that for some constant $M\ge 1$

$$\|\lambda J_\lambda f\|_\infty\le M\|f\|_\infty \text{ for all } \lambda\in D_+ \text{ and } f\in L_1\cap L_\infty,$$

then the following individual ergodic limits

$$\lim_{\substack{\lambda\to 0\\ \lambda\in D_+}}\lambda J_\lambda f(\omega) \quad\text{and}\quad \lim_{\substack{\lambda\to\infty\\ \lambda\in D_+}}\lambda J_\lambda f(\omega)$$

exist almost everywhere, whenever $f\in L_p$ with $1\le p<\infty$.

The purpose of this note is to examine the necessity of the above norm conditions on J and to show, by examples, that these conditions can not be weakened without failing to hold the individual ergodic theorem.

2. Preliminary lemmas.

Lemma 1. *If $(a_n:n\ge 0)$ is a sequence of nonnegative reals, then*

$$\sup_{n\ge 1}\frac{1}{n}\sum_{i=0}^{n-1}a_i\le e\cdot\sup_{0<r<1}(1-r)\sum_{i=0}^{\infty}r^ia_i$$

and

$$\limsup_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1}a_i\le e\cdot\limsup_{r\to 1-0}(1-r)\sum_{i=0}^{\infty}r^ia_i.$$

Proof. Putting

$$C_n=\frac{1}{n}\sum_{i=0}^{n-1}a_i\quad(n\ge 1)\quad\text{and}\quad A_r=(1-r)\sum_{i=0}^{\infty}r^ia_i\quad(0<r<1),$$

we see that

$$A_r\ge(1-r)\sum_{i=0}^{n-1}r^ia_i\ge(1-r)r^{n-1}\sum_{i=0}^{n-1}a_i=n(1-r)r^{n-1}C_n.$$

On the other hand, for each $n \geq 1$

$$\sup_{0<r<1} n(1-r)r^{n-1} = n(1-[1-\frac{1}{n}])(1-\frac{1}{n})^{n-1} > e^{-1};$$

therefore the lemma follows.

Lemma 2. *If $a(t)$ is a nonnegative Lebesgue measurable function on the interval $(0, \infty)$, then*

$$\sup_{b>0} \frac{1}{b}\int_0^b a(t)\ dt \leq e\cdot\sup_{\lambda>0} \lambda\int_0^\infty e^{-\lambda t}a(t)\ dt,$$

$$\limsup_{b\to+\infty} \frac{1}{b}\int_0^b a(t)\ dt \leq e\cdot\limsup_{\lambda\to+0} \lambda\int_0^\infty e^{-\lambda t}a(t)\ dt$$

and

$$\limsup_{b\to+0} \frac{1}{b}\int_0^b a(t)\ dt \leq e\cdot\limsup_{\lambda\to+\infty} \lambda\int_0^\infty e^{-\lambda t}a(t)\ dt.$$

Proof. Since

$$\lambda\int_0^\infty e^{-\lambda t}a(t)\ dt \geq \lambda e^{-\lambda b}\int_0^b a(t)\ dt$$
$$= (\lambda b)e^{-\lambda b}\cdot\frac{1}{b}\int_0^b a(t)\ dt,$$

and since for each $b > 0$

$$\sup_{\lambda>0} (\lambda b)e^{-\lambda b} = e^{-1},$$

the lemma follows immediately.

3. Counterexamples.

Example 1 (cf. [7]). *There exists a pseudo-resolvent $\boldsymbol{J} = (J_\lambda : \lambda\in D)$ on $L_1(0, 1)$ such that*

(i) *for all $\lambda\in D_+$, $J_\lambda \geq 0$ and $\|\lambda J_\lambda\|_1 = 1$,*

(ii) *for some $f\in L_1(0, 1)$ the limit*

$$\lim_{\substack{\lambda\to 0\\ \lambda\in D_+}} \lambda J_\lambda f(\omega)$$

does not exist almost everywhere on the whole interval $(0, 1)$.

To see this, let T be a positive isometry on $L_1(0,1)$ such that for some $0\leq f\in L_1(0, 1)$, $\limsup_n \frac{1}{n}\sum_{i=0}^{n-1} T^i f(\omega) = \infty$ almost everywhere on $(0, 1)$ (cf. [1]). Then, since $\|T\|_1 = 1$, we may define

$$J_\lambda = (\lambda+1-T)^{-1} = \frac{1}{\lambda+1}\sum_{i=0}^{\infty}\frac{T^i}{(\lambda+1)^i} \quad (\lambda \in D).$$

Clearly, $J = (J_\lambda : \lambda \in D)$ is a pseudo-resolvent on $L_1(0, 1)$ such that

$$J_\lambda \geq 0 \text{ and } \|\lambda J_\lambda\|_1 = 1 \text{ for all } \lambda \in D_+.$$

Furthermore, by Lemma 1, we have

$$\limsup_{\substack{\lambda\to 0\\ \lambda\in D_+}} \lambda J_\lambda f(\omega) = \infty$$

almost everywhere on (0, 1). On the other hand, by Fatou's lemma, if we set

$$h(\omega) = \liminf_{\substack{\lambda\to 0\\ \lambda\in D_+}} \lambda J_\lambda f(\omega) \quad \text{for all } \omega \in (0, 1)$$

then $0 \leq h \in L_1(0, 1)$. Therefore $h(\omega) < \infty$ almost everywhere on (0, 1), and this completes the proof.

Example 2. *Given an $\varepsilon > 0$ there exists a pseude-resolvent $J = (J_\lambda : \lambda \in D)$ on L_1 of a finite measure space such that*

(i) *for all $\lambda \in D_+$*

$$J_\lambda \geq 0,\ \lambda J_\lambda 1 = 1 \ \textit{and}\ \|\lambda J_\lambda\|_1 \leq 1+\varepsilon,$$

(ii) *for some $f \in L_1$ the limit*

$$\lim_{\substack{\lambda\to 0\\ \lambda\in D_+}} \lambda J_\lambda f(\omega)$$

does not exist almost everywhere on a certain measurable subset of positive measure.

To see this, let S be an ergodic and invertible measure preserving point transformation on the interval (0, 1] and define also $Sg(\omega) = g(S\omega)$ for $g \in L_1(0, 1]$. Take $0 \leq f \in L_1(0, 1]$ such that $f \log^+ f \notin L_1(0, 1]$. Then, by [3] and Lemma 1, we see that

$$\sup_{0<r<1} (1-r)\sum_{i=0}^{\infty} r^i S^i f(\omega) \notin L_1(0, 1].$$

Thus, as in Derriennic and Lin [2], there exists a sub-σ-field $\mathcal{B}$ of the Lebesgue measurable subsets of (0, 1] such that the limit

$$\lim_{r\to 1-0} (1-r) \sum_{i=0}^{\infty} r^i E[S^i f \mid \mathcal{B}](\omega)$$

does not exist almost everywhere on (0, 1], where $E\,[\cdot \mid \mathcal{B}]$ stands for the conditional expectation operator with respect to $\mathcal{B}$. Define a positive linear operator T on $L_1(0, 1+\varepsilon]$ by

$$Tg(\omega) = \begin{cases} S(g1_{(0,1]})(\omega) & \text{if } 0 < \omega \leq 1, \\ E[S(g1_{(0,1]}) \mid \mathcal{B}](\frac{\omega-1}{\varepsilon}) & \text{if } 1 < \omega \leq 1+\varepsilon. \end{cases}$$

It is easily seen that $T1 = 1$ and $\|T^n\|_1 = 1+\varepsilon$ $(n \geq 1)$. Thus if we set

$$J_\lambda = (\lambda+1-T)^{-1} = \frac{1}{\lambda+1}\sum_{i=0}^{\infty}\frac{T^i}{(\lambda+1)^i} \quad (\lambda \in D)$$

then for all $\lambda \in D_+$ we have

$$J_\lambda \geq 0, \ \lambda J_\lambda 1 = 1 \text{ and } \|\lambda J_\lambda\|_1 \leq 1+\varepsilon\,;$$

furthermore, $\lim_{\substack{\lambda\to 0\\ \lambda\in D_+}} \lambda J_\lambda f(\omega)$ does not exist almost everywhere on $(1, 1+\varepsilon]$. Hence the proof is completed.

Example 3. *Given an $\varepsilon > 0$ there exists a pseudo-resolvent $\boldsymbol{J} = (J_\lambda : \lambda \in D)$ on L_1 of a finite measure space such that*

(i) *for all $\lambda \in D_+$*

$$J_\lambda \geq 0, \ \lambda J_\lambda 1 = 1 \text{ and } \|\lambda J_\lambda\|_1 \leq 1+\varepsilon,$$

(ii) *for some $f \in L_1$ the limit*

$$\lim_{\substack{\lambda\to\infty\\ \lambda\in D_+}} \lambda J_\lambda f(\omega)$$

does not exist almost everywhere on a certain measurable subset of positive measure.

To see this, let $(S_t : t \geq 0)$ be the strongly continuous semigroup of positive isometries on $L_1(0, 1]$ defined by

$$S_t g(\omega) = g(\omega \dot{+} t) \qquad (g \in L_1(0, 1],\ \omega \in (0, 1]),$$

where $\omega \dot{+} t = \omega + t$ if $\omega + t \leq 1$ and $\omega \dot{+} t = \omega + t - n$ if $n < \omega + t \leq n+1$. By [4], for some $0 \leq f \in L_1(0, 1]$ and some sequence (b_n) of positive reals with $\lim_n b_n = 0$ we have

$$\sup_n \frac{1}{b_n}\int_0^{b_n} S_t f(\omega)\, dt \notin L_1(0, 1).$$

Thus, by Lemma 2 and the argument given in Example 2 (cf. also [4]), there exists a strongly continuous semigroup $\Gamma = (T_t : t \geq 0)$ of positive linear operators on $L_1(0, 1+\varepsilon]$ such that for all $t \geq 0$

$$T_t 1 = 1 \text{ and } \|T_t\|_1 = 1+\varepsilon,$$

and also such that the limit

$$\lim_{\substack{\lambda\to\infty \\ \lambda\in D_+}} \lambda\int_0^\infty e^{-\lambda t}\, T_t f(\omega)\, dt$$

does not exist almost everywhere on $(1, 1+\varepsilon]$. For $\lambda \in D$ define

$$J_\lambda g = \int_0^\infty e^{-\lambda t}\, T_t g\, dt \quad (g \in L_1(0, 1+\varepsilon]).$$

Obviously J_λ is a bounded linear operator on $L_1(0, 1+\varepsilon]$ satisfying $J_\lambda T_0 = J_\lambda$, and if $\lambda \in D_+$ then $J_\lambda \geq 0$ and $\|\lambda J_\lambda\|_1 = 1+\varepsilon$. Thus, to complete the proof it is now enough to check that $J = (J_\lambda : \lambda \in D)$ is a pseudo-resolvent. To this end, put $M = T_0 L_1(0, 1+\varepsilon]$. Then M is a closed subspace of $L_1(0, 1+\varepsilon]$, $T_t M \subset M$ for all $t \geq 0$, and $T_0 = I$ on M. Thus, by restricting $\Gamma = (T_t : t \geq 0)$ to M and applying Corollary IX.4.1 and Theorem VIII.2.2 in [6], we see that $J_\lambda - J_\nu = (\nu-\lambda) J_\lambda J_\nu$ on M. Hence for every $g \in L_1(0, 1+\varepsilon]$

$$\begin{aligned} J_\lambda g - J_\nu g &= J_\lambda T_0 g - J_\nu T_0 g \\ &= (\nu-\lambda) J_\lambda J_\nu T_0 g = (\nu-\lambda) J_\lambda J_\nu g, \end{aligned}$$

completing the proof.

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