Math. J. Okayama Univ. 23 (1981), 189—193

ON THE RADICAL OF THE GROUP ALGEBRA OF A p-NILPOTENT GROUP

YASUSHI NINOMIYA

Throughout the present paper, K will represent an algebraically closed field of characteristic $p>0$, and G a finite group whose order is divisible by p. We denote by $J(KG)$ the radical of the group algebra KG. In the previous paper [4], we proved that $J(KG)$ is contained in $J(KP)KG$ for some Sylow p-subgroup P of G if and only if $J(KG)=\bigcap_{x\in G}J(KP^{x})KG$ ([4, Theorem 3]). For convenience' sake, we denote by $\mathfrak{P}$ the class of finite groups G such that $J(KG)=\bigcap J(KP)KG$, where P ranges over Sylow p-subgroups of G. In [4], we studied the properties of groups contained in $\mathfrak{P}$. Recently, S. S. Bedi [1] gave several sufficient conditions for a group to be contained in $\mathfrak{P}$, but all of his results had been obtained in [4]. The purpose of this paper is to give a necessary and sufficient condition for a p-nilpotent group to be in $\mathfrak{P}$. Given $g\in G$, we put $a^{g}=gag^{-1}$ for any $a\in KG$, and $S^{g}=\{s^{g}|s\in S\}$ for any subset S of KG. We denote by E_G the set of all central primitive idempotents of KG.

In what follows, we let G be a p-nilpotent group, and $N=O_{p'}(G)$. Let $f\in E_N$. Then fKN is isomorphic to the matrix ring $(K)_n$ over K. We put $G_f=\{g\in G|f^{g}=f\}$, and denote by P_f a Sylow p-subgroup of G_f. Now, let $G=a_1G_f\cup a_2G_f\cup\cdots\cup a_sG_f$ be the decomposition of G into right cosets with respect to G_f. Then by [3], $e_f=\sum_{i=1}^{s}f^{a_i}\in E_G$ and e_fKG is isomorphic to the matrix ring $(KP_f)_{ns}$ over KP_f.

Lemma. *If G is in $\mathfrak{P}$ then every normal subgroup H of G is in $\mathfrak{P}$.*

Proof. Since H is a normal subgroup of a p-nilpotent group G, we have $J(KH)=KH\cap J(KG)\subset KH\cap J(KP)KG=J(KQ)KH$, where P is a Sylow p-subgroup of G and $Q=H\cap P$. Hence $H\in\mathfrak{P}$.

Now, we can state our theorem as follows:

Theorem. *The following statements are equivalent:*

(1) *G is in $\mathfrak{P}$.*

(2) *If $f\in E_N$, then $fx=f$ for every $x\in[N, P_f]$.*

(3) *If $f\in E_N$, then every element of P_f commutes with all the elements of fKN.*

Proof. (1) $\Longrightarrow$ (2): We may assume that $P_f \neq \{1\}$. Since G_f is a subnormal subgroup of G, $G_f \in \mathfrak{P}$ by Lemma. Hence, by [4, Theoren 3], $J(fKG_f) = fJ(KG_f) \subset fJ(KP_f^g)KG_f = J(KP_f^g)(fKN)$, where $g \in G_f$. Now, let $fKN \cong (K)_n$. Then we have $fKG_f \cong (KP_f)_n$. Since $\dim_K J(fKG_f) = \dim_K (J(KP_f))_n = (|P_f|-1)n^2 = \dim_K J(KP_f) \cdot \dim_K fKN = \dim_K J(KP_f^g) \cdot \dim_K fKN = \dim_K J(KP_f^g)(fKN)$, the above implies that $J(fKG_f) = J(KP_f^g)(fKN)$. Let $s \in P_f$ and $x \in N$. Noting that $J(KP_f)(fKN) = J(KP_f^x)(fKN)$, we have $0 = \widehat{P}_f(s^x - 1)f = \widehat{P}_f(s^{-1}s^x - 1)f$, where $\widehat{P}_f = \sum_{u \in P_f} u$. This implies that $f(s^{-1}s^x) = (s^{-1}s^x)f = f$, and so (2) holds.

(2) $\Longrightarrow$ (3): Let $s \in P_f$ and $x \in N$. Then we have $s(fx)s^{-1} = fsxs^{-1} = fx(x^{-1}sxs^{-1}) = x(fx^{-1}sxs^{-1}) = xf = fx$, proving (3).

(3) $\Longrightarrow$ (1): Since every element of P_f commutes with all the elements of fKN, fKG_f is a group ring of P_f over the simple ring fKN. Hence we have $fJ(KG_f) = (fKN)J(KP_f)$. Furthermore, by [5, Theorem 5], we see that $(fKG_f/fJ(KG_f)) \otimes_{KG_f} KG$ is a completely reducible KG-module. Since $(fKG_f/fJ(KG_f)) \otimes_{KG_f} KG \cong fKG/fJ(KG_f)KG$, this implies that $fJ(KG) \subset fJ(KG_f)KG = (fKN)J(KP_f)KG = fJ(KP_f)KG \subset J(KP_f)KG$. Now, let P be a Sylow p-subgroup of G which contains P_f. Then $J(KP_f) \subset J(KP)$, and so $fJ(KG) \subset J(KP)KG$, proving (1).

Corollary 1. *If N is abelian then G is in $\mathfrak{P}$.*

Proof. Since $fKN = fK$ for every $f \in E_N$, it is clear that $xax^{-1} = a$ for every $a \in fKN$ and $x \in G_f$. Hence $G \in \mathfrak{P}$ by Theorem.

Corollary 2. *Let P be a Sylow p-subgroup of G. Assume that $P \cap P^x = \{1\}$ for every $x \in G - N_G(P)$. Then G is in $\mathfrak{P}$ if and only if there holds one of the following:*

(1) *P is normal in G.*

(2) *G has a subnormal subgroup H which is a Frobenius group with complement P.*

Proof. Suppose that G is in $\mathfrak{P}$ and P is not normal in G. Then, we may assume that G has no normal subgroups of index relatively prime to p. Since $[N, P]P$ is normal in G, we have $[N, P] = N$. Let $f \in E_N$ and suppose that $P_f \neq \{1\}$. Since P_f is a defect group of $e_f KG$, our assumption together with [2, Theorem 2] implies that P_f is a Sylow p-subgroup of G. So we may assume that $P_f = P$. Then, by Theorem (2), we have $fx = f$ for every $x \in [N, P] = N$, and therefore $f = |N|^{-1}\sum_{x \in N} x$. Thus, we see that every block of KG different from the principal block is of defect zero.

Since $N=[N, P]\neq\{1\}$, by [6, Theorem 2], G is a Frobenius group with complement P.

Conversely, if P is normal in G then $J(KG)=J(KP)KG$, and so $G\in\mathfrak{P}$. Next, suppose that (2) holds, and let V be the Frobenius kernel of H. Putting $e=|V|^{-1}\sum_{v\in V} v$, we obtain $J(KG)=J(KH)KG=eJ(KP)KG\subset J(KP)KG$. Hence $G\in\mathfrak{P}$.

Corollary 3. *Suppose that G is in $\mathfrak{P}$ and a Sylow p-subgroup P of G is a cyclic group of order p^a generated by s. Let $D_i=\langle s^{p^i}\rangle$, $V_i=[N, D_i]$ and $e_i=|V_i|^{-1}\sum_{x\in V_i}x$, where $0\leqq i\leqq a-1$. Then, e_0 is the sum of block idempotents of defect a, and $e_i-e_{i-1}(1\leqq i\leqq a-1)$ is the sum of block idempotents of defect $a-i$. In particular, the sum of block ideals of positive defect is isomorphic to KG/V_{a-1}.*

Proof. By Theorem (2), we see that a block ideal of defect $a-i$ is contained in e_iKG. Since $e_iKG\cong KG/V_i$ and D_iV_i/V_i is normal in G/V_i, e_iKG is the sum of block ideals of defect $\geqq a-i$. Noting that $e_iKG=e_{i-1}KG\oplus(e_i-e_{i-1})KG$, we can easily see that the result holds.

A. I. Saksonov and D. S. Passman individually gave examples of $\mathfrak{P}$ p-nilpotent groups G such that $G\notin\mathfrak{P}$ (see [4] and [1]). Now, by making use of Corollary 3, we shall show that these groups are not in $\mathfrak{P}$.

Example 1 (Saksonov). Let $p=3$, and $G=SL(2,3)$. Then G is a 3-nilpotent group with a cyclic Sylow 3-subgroup P of order 3. Suppose $G\in\mathfrak{P}$. Since $[O_{3'}(G), P]=O_{3'}(G)$, by Corollary 3 we have $J(KG)=eJ(KP)$, where $e=|O_{3'}(G)|^{-1}\sum_{x\in O_{3'}(G)}x$. Hence, by [6, Theorem 2], G is a Frobenius group with complement P. But, this is a contradiction, because G has the non-trivial center. Hence $G\notin\mathfrak{P}$.

Example 2 (Passman). Let $p=2$. Obviously,

$$N=\left\{\begin{pmatrix}1 & \alpha & \beta\\ 0 & 1 & \gamma\\ 0 & 0 & 1\end{pmatrix}\middle|\ \alpha, \beta, \gamma\in GF(3)\right\}$$

is a group of order 27. Let $p=\langle s\rangle$ be a group of order 2, and G a semi-direct product of N by P, where the action of s to N is defined as follows:

$$\begin{pmatrix}1 & \alpha & \beta\\ 0 & 1 & \gamma\\ 0 & 0 & 1\end{pmatrix}^s=\begin{pmatrix}1 & -\alpha & \beta\\ 0 & 1 & -\gamma\\ 0 & 0 & 1\end{pmatrix}.$$

Then G is a 2-nilpotent group. Consider

$$a=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\quad b=\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}.$$

Since $c=aba^{-1}b^{-1}=\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$ is an element of the center of N, we see that N is generated by a and b. Further, noting that $sas^{-1}a^{-1}=a$ and $sbs^{-1}b^{-1}=b$, we have $[N, P]=N$. Suppose $G\in\mathfrak{P}$. Then Corollary 3 together with the fact above implies that $J(KG)=eJ(KP)$, where $e=|N|^{-1}\sum_{x\in N}x$. Hence by [6, Theorem 2], G is a Frobenius group with complement P. But this is a contradiction, because c is contained in the center of G. Hence $G\notin\mathfrak{P}$.

Let G be an arbitrary finite group (not necessarily a p-nilpotent group). In [1], S. S. Bedi asked: Does every G in $\mathfrak{P}$ have a normal subgroup G_0 such that $p \nmid [G : G_0]$ and that the factor group of G_0 by some normal p-subgroup is a Frobenius group with a Sylow p-subgroup as a complement? The next example gives a negative answer to this question.

Example 3. Let $p=2$. Let N be an elementary abelian group of order 9 generated by b_1 and b_2, and

$$P=\langle s,\ t\,|\,s^4=1,\ t^2=1,\ tst^{-1}=s^{-1}\rangle$$

a dihedral group of order 8. We define a homomorphism $\theta: P\longrightarrow GL(2,3)$ ($\cong$ Aut N) by

$$\theta(s)=\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad \theta(t)=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

Now, let G be a semi-direct product of N by P with respect to θ. Since N is abelian, G is in $\mathfrak{P}$ by Corollary 1. However, G does not satisfy the condition in the above question. In fact, G has no normal 2-subgroups and the dihedral group cannot be a Frobenius complement.

References

[1] S. S. Bedi: The Jacobson radical of the group algebra of a finite group, Proc. Amer. Math. Soc. 75 (1979), 13—18.

[2] J. A. Green: Blocks of modular representations, Math. Z. 79 (1962), 100—115.

[3] K. Morita: On group rings over a modular field which possess radicals expressible as

principal ideals, Sci. Rep. Tokyo Bunrika Daigaku **A4** (1951), 177—194.
[4] K. MOTOSE and Y. NINOMIYA: On the subgroups H of a group G such that $J(KH)KG \supset J(KG)$, Math. J. Okayama Univ. **17** (1975), 171—176.
[5] M. OSIMA: On primary decomposable group rings, Proc. Phys.-Math. Soc. Japan **24** (1942), 1—9.
[6] D. A. R. WALLACE: Note on the radical of a group algebra, Proc. Cambridge Philos. Soc. **54** (1958), 128—130.

SHINSHU UNIVERSITY

(*Received May 7, 1981*)