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NOTE ON COMMUTATIVITY OF RINGS. III

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Throughout, R will represent an associative ring with center C, J the (Jacobson) radical of R, D the commutator ideal of R, and N the set of all positive integers. A ring R is called *s-unital* if for each $x \in R$, $x \in Rx \cap xR$. As stated in [1, Lemma 1 (a)], if R is an s-unital ring, then for any finite subset F of R, there exists an element e in R such that $ex = xe = x$ for all x in F. Such an element e will be called a *pseudo-identity* of F.

We consider the following properties of rings:

(I) For each pair of elements x, y in R, there exist positive integers m, m' such that $(m, m') = 2$ and

$$(xy)^a = x^a y^a, \quad a = m,\ m+1,\ m',\ m'+1.$$

(II) For each pair of elements x, y in R there exists an even positive integer m such that

$$(xy)^a = x^a y^a, \quad a = m,\ m+2,\ m+4.$$

The purpose of this note is to prove the following commutativity theorems.

Theorem 1. *If R is an s-unital ring having the property* (I), *then R is commutative.*

Theorem 2. *If R is an s-unital ring having the property* (II), *then R is commutative.*

Recently, C.-T. Yen [3] showed that every primary ring satisfying the polynomial identities $(xy)^a = x^a y^a$ $(a = m, m+1, m', m'+1)$ with $(m, m') = 1$ or 2 is commutative. Obviously, Theorem 1 together with the previous result [2, Theorem] improves the result of Yen.

In preparation for the proof of our theorems, we first state an easy lemma.

Lemma 1. (1) *Let e be a pseudo-identity of $a \in R$. If a has a quasi-inverse a', then $(a+e)(a'+e) = (a'+e)(a+e) = e^2$.*

(2) *Let e be a pseudo-identity of $\{a, b\} \subseteq R$. If $a^s b = 0 = (a+e)^t b$ for some s, $t \in N$, then $b = 0$.*

(3) *Let* $m \in 2N$, *and* $a, b \in R$. *If* $[a^2, b^2]=0$ *and* $(ab)^\alpha = a^\alpha b^\alpha$ *for* $\alpha = m, m+2$, *then* $b^m a^{m+1}[b,a]b = 0$.

Proof. (1) is immediate and (2) is well known.

(3) In fact,

$$0 = (ab)^{m+2} - a^{m+2}b^{m+2} = a^m b^m (ab)^2 - a^{m+2}b^{m+2}$$
$$= b^m a^{m+1} bab - b^m a^{m+2} b^2 = b^m a^{m+1}[b,a]b.$$

Corollary 1. *Let* e *be a pseudo-identity of* $\{a, b\} \subseteq R$, *and* $m \in 2N$. *Suppose that* $\{b(a+e)\}^\alpha = b^\alpha (a+e)^\alpha$ *for* $\alpha = m, m+2, m+4$. *If* a *is quasi-regular, then* $b^{m+2}[b^2, (a+e)^2] = 0$.

Proof. By Lemma 1 (1), we can easily see that

$$\{b(a+e)\}^2 b^m = b^{m+2}(a+e)^2 \text{ and } \{b(a+e)\}^2 b^{m+2} = b^{m+4}(a+e)^2.$$

Combining those above, we readily obtain $b^{m+2}[b^2, (a+e)^2] = 0$.

In order to prove Theorem 1, we require further lemmas.

Lemma 2. *Let* R *be an s-unital ring having the property* (I). *Then, for* $a, b \in R$, *there exists* $s \in N$ *such that* $[a, b^2]b^s = 0$.

Proof. By hypothesis, there exist $m, m' \in N$ with $(m, m') = 2$ such that

$$(ab)^\alpha = a^\alpha b^\alpha, \quad \alpha = m, m+1, m', m'+1.$$

Without loss of generality, we may assume that $t'm' - tm = 2$ with some $t, t' \in N$. By [2, Lemma 1 and Lemma 2 (a)], there exist $p, p' \in N$ such that

$$[a, b^{tm}]b^2 a^p = 0 \text{ and } [a, b^{t'm'}]b^2 a^{p'} = 0.$$

Putting $p'' = \max\{p, p'\}$, we obtain

$$[a,b^2]b^{t'm'}a^{p''} = [a,b^2]b^{tm}b^2 a^{p''} = [a,b^{t'm'}]b^2 a^{p''} - b^2[a,b^{tm}]b^2 a^{p''} = 0.$$

Again by [1, Lemma 1], there exists $q \in N$ such that $a^q[a,b^2]b^{t'm'} = 0$. Hence, by Lemma 1 (2), $[a,b^2]b^s = 0$ with some $s \in N$.

Lemma 3. *Let* R *be an s-unital ring having the property* (I). *If* a *is a quasi-regular element in* R, *then* a *is central. In particular, every division ring having the property* (I) *is commutative.*

Proof. Let $b \in R$, and e a pseudo-identity of $\{a, b\}$. Then, by Lemma 2,

$[b,(a+e)^2](a+e)^s = 0$ for some $s \in N$, and therefore $[b,(a+e)^2] = 0$ by Lemma 1 (1). Choose $m, m' \in N$ with $(m, m') = 2$ such that

$$\{b(a+e)\}^\alpha = b^\alpha(a+e)^\alpha, \quad \alpha = m, m+1, m', m'+1.$$

Without loss of generality, we may assume that $t'm' - tm = 2$ with some $t, t' \in N$. Then, noting that $[b,(a+e)^2] = 0$ and both m and m' are multiples of 2, we can easily see that

$$\{b(a+e)\}^\alpha = b^\alpha(a+e)^\alpha, \quad \alpha = tm, tm+1, tm+2 = t'm'.$$

From those above, it follows that $b[b^{tm},a] = b[b^{tm},a+e] = 0$ and $b[b^{tm+1},a] = b[b^{tm+1},a+e] = 0$. Hence, $b^{tm+1}[b,a] = b[b^{tm+1},a] - b[b^{tm},a]b = 0$. Now, by Lemma 1 (2), we obtain $[b,a] = 0$.

Corollary 2. *If R is an s-unital ring having the property* (I), *then* $D \subseteq J \subseteq C$.

Proof. Since $J \subseteq C$ by Lemma 3, it remains only to prove that $D \subseteq J$. Note that the property (I) is inherited by all subrings and homomorphic images of R. Note also that no complete matrix ring $(S)_t$ over a division ring S $(t > 1)$ has the property, as a consideration of $x = E_{12}$ and $y = E_{21}$ shows. It suffices to show that if R is a semi-primitive s-unital ring having the property (I) then it is commutative. Because of the above facts and the structure theory of primitive rings, we may assume that R is a division ring. Then, R is commutative by Lemma 3.

Lemma 4. *Let R be an s-unial ring having the propecty* (I). *If* $2[a,b] = 0$ then $[a,b] = 0$.

Proof. Since $D \subseteq C$ by Corollary 2, we have $[a^2,b] = 2a[a,b] = 0$. Choose $m, m' \in N$ with $(m, m') = 2$ such that

$$(ab)^\alpha = a^\alpha b^\alpha, \quad \alpha = m, m+1, m', m'+1.$$

Without loss of generality, we may assume that $t'm' - tm = 2$ with some $t, t' \in N$. Then, noting that $[a^2,b] = 0$ and both m and m' are multiples of 2, we can easily see that

$$(ab)^\alpha = a^\alpha b^\alpha, \quad \alpha = mt, mt+1, mt+2 = m't'.$$

Now, applying the same argument as in the last part of the proof of Lemma 3, we obtain $[a,b] = 0$.

We are now in a position to complete the proof of Theorem 1.

Proof of Theorem 1. Let a, b be arbitrary elements of R. According to Lemma 2, there exists $s \in N$ such that $[a,b^2]b^s = 0$. Since $[a,b]$ is central by Corollary 2, $2[a,b]b^{s+1} = [a,b^2]b^s = 0$. Hence, $2[a,b] = 0$ by Lemma 1 (2), and so $[a,b] = 0$ by Lemma 4.

Our next task is to prove Theorem 2. To this end, we state the following lemmas.

Lemma 5. *Let R be an s-unital ring having the property* (Π), *and* a, $b \in R$. *If a is quasi-regular and* $2[a,b] = 0$, *then* $[a,b] = 0$.

Proof. Obviously, $2[a,b] = 0$ implies $2[b^2,a] = 0 = 2[b,a^2]$. Let e be a pseudo-identity of $\{a, b\}$, and e' a pseudo-identity of $\{a, b, e\}$. By hypothesis, there exist m, $n \in 2N$ such that

$$\{b(a+e)\}^\alpha = b^\alpha(a+e)^\alpha,\quad \alpha = m,\ m+2,\ m+4, \text{ and}$$
$$\{(b+e)(a+e')\}^\beta = (b+e)^\beta(a+e')^\beta,\quad \beta = n,\ n+2,\ n+4.$$

Then, by Corollary 1, we see that

$$b^{m+2}[b^2,a^2] = b^{m+2}[b^2,(a+e)^2] = 0,$$
$$(b+e)^{n+2}[b^2,a^2] = (b+e)^{n+2}[(b+e)^2,a^2] = 0.$$

Hence, by Lemma 1 (2), $[b^2,a^2] = 0$, and so $[b^2,(a+e)^2] = 0$. Now, according to Lemma 1 (3), we get $(a+e)^m b^{m+1}[a+e,b](a+e) = 0$, and so $b^{m+1}[a,b] = 0$ by Lemma 1 (1). Similarly, $(b+e)^{n+1}[a,b] = 0$. Thus, again by Lemma 1 (2), $[a,b] = 0$.

Lemma 6. *Let R be an s-unital ring having the property* (Π). *If a and b are quasi-regular elements of R, then* $[a,b] = 0$. *In particular, every division ring having the property* (Π) *is commutative.*

Proof. Let e be a pseudo-identity of $\{a, b\}$. By hypothesis, there exists $m \in 2N$ such that

$$\{(a+e)(b+e)\}^\alpha = (a+e)^\alpha(b+e)^\alpha,\quad \alpha = m,\ m+2,\ m+4.$$

Then, by making use of the argument used in the proof of Corollary 1, we can easily see that $e^{2(m+2)}(a+e)^{m+2}[(b+e)^2,(a+e)^2] = 0$. Since $[(b+e)^2, (a+e)^2] = [b^2+2b, a^2+2a]$, this together with Lemma 1 (1) implies $[(a+e)^2,(b+e)^2] = 0$. Hence, by Lemma 1 (3), $(b+e)^m(a+e)^{m+1}[b,a](b+e) = 0$. Now, again by Lemma 1 (1), we get $[b,a] = 0$.

Lemma 7. *If R is an s-unital ring having the proerty* (Π), *then* $D \subseteq J \subseteq C$.

Proof. By Lemma 6, every division ring having the property (II) is commutative. Hence, applying the argument used in the proof of Corollary 2, we can see that $D \subseteq J$. It remains therefore to show that $J \subseteq C$. Now, let a be in J, and b an arbitrary element of R. Let e be a pseudo-identity of $\{a, b\}$, and e' a pseudo-identity of $\{a, b, e\}$. By hypothesis, there exist $m, n \in 2N$ such that

$$\{b(a+e)\}^{\alpha} = b^{\alpha}(a+e)^{\alpha}, \quad \alpha = m, m+2, m+4,$$
$$\{(b+e)(a+e')\}^{\beta} = (b+e)^{\beta}(a+e')^{\beta}, \quad \beta = n, n+2, n+4.$$

Then, by Corollary 1,

$$b^{m+2}[b^2, a^2+2a] = b^{m+2}[b^2, (a+e)^2] = 0, \text{ and}$$
$$(b+e)^{n+2}[b^2+2b, a^2+2a] = (b+e)^{n+2}[(b+e)^2, a^2+2a] = 0.$$

From these, it follows that

$$2b^{m+2}(b+e)^{n+2}[b, a^2+2a] = (b+e)^{n+2}b^{m+2}[b^2+2b, a^2+2a] = 0.$$

Similarly, we see that there exists $n' \in 2N$ such that

$$2(b+e)^{n+2}(b+e+e')^{n'+2}[b, a^2+2a] = 0.$$

Hence,

$$2^{n'+3}b^{m+1}(b+e)^{n+2}[b, a^2+2a] = 2(b+e)^{n+2}b^{m+1}(b+e+e')^{n'+2}[b, a^2+2a] = 0.$$

Continuing this procedure, we obtain eventually

$$2^k(b+e)^{n+2}[b+e, a^2+2a] = 2^k(b+e)^{n+2}[b, a^2+2a] = 0, \text{ and}$$
$$2^k(b+e+e')^{n'+2}[b+e, a^2+2a] = 0$$

for some $k \in N$. From these, it follows that $2^k[b, a^2+2a] = 0$ (Lemma 1 (2)). Then, since $a^2+2a \in J$, we have $[b, (a+e)^2] = [b, a^2+2a] = 0$ (Lemma 5), and so

$$(a+e)^m b^{m+1}[a, b](a+e) = (a+e)^m b^{m+1}[a+e, b](a+e) = 0$$

by Lemma 1 (3). Hence, $b^{m+1}[a, b] = 0$ by Lemma 1 (1). Now, $[a, b] = 0$ is immediate by Lemma 1 (2).

Corollary 3. *Let R be an s-unital ring having the property* (II). *If* $2[a, b] = 0$ *then* $[a, b] = 0$.

Proof. Since $[a, b] \in C$ (Lemma 7) and $[a, b^2] = 2b[a, b] = 0$, we have $a^{m+1}[b, a]b^{m+1} = b^m a^{m+1}[b, a]b = 0$ for some $m \in 2N$ (Lemma 1 (3)). Thus, $[b, a] = 0$ is an easy consequence of Lemma 1 (2).

We are now ready to complete the proof of Theorem 2.

Proof of Theorem 2. Let a and b be arbitrary elements of R. By (II), there exists $m \in 2N$ such that

$$(ab)^\alpha = a^\alpha b^\alpha, \quad \alpha = m, \quad m+2, \ m+4.$$

Then,

$$[(ab)^2, a^m]b^m + a^m\{(ab)^2 - a^2b^2\}b^m = \{(ab)^2a^m - a^{m+2}b^2\}b^m$$
$$= (ab)^2a^mb^m - a^{m+2}b^{m+2} = 0$$

and

$$[(ab)^2, a^{m+2}]b^{m+2} + a^{m+2}\{(ab)^2 - a^2b^2\}b^{m+2} = 0.$$

From those above, we readily obtain

$$a^2[(ab)^2, \ a^m]b^{m+2} - [(ab)^2, a^{m+2}]b^{m+2} = 0.$$

Hence, noting that $D \subseteq C$ (Lemma 7), we see that

$$4a^{m+3}[a,b]b^{m+3} = 4a^{m+1}[a,ab](ab)b^{m+2} = 2a^{m+1}[a,(ab)^2]b^{m+2}$$
$$= ma^{m+1}[(ab)^2,a]b^{m+2} - (m+2)a^{m+1}[(ab)^2, a]b^{m+2}$$
$$= a^2[(ab)^2,a^m]b^{m+2} - [(ab)^2, a^{m+2}]b^{m+2} = 0.$$

Now, by Lemma 1 (2) and Corollary 3, it is easy to see that $[a,b] = 0$.

Remark. In Theorem 1 and Theorem 2, we can replace (I) and (II) by the following properties, respectively:

(I)′ For each pair of elements x, y in R, there exist positive integers m, m' such that $(m+1, \ m'+1) = 2$ and

$$(xy)^\alpha = y^\alpha x^\alpha, \quad \alpha = m, \ m+1, \ m', \ m'+1.$$

(II)′ For each pair of elements x, y in R, there exists an odd positive integer m such that

$$(xy)^\alpha = y^\alpha x^\alpha, \quad \alpha = m, \ m+2, \ m+4.$$

Furthermore, careful examination of the proofs of Lemmas 1 (3), 5, 6, 7, Corollaries 1, 3, and of Theorem 2 shows that Theorem 2 is still true if (II) is replaced by the following property:

(II)″ For each pair of elements x, y in R, there exists an even positive integer m such that

$$(xy)^\alpha = y^\alpha x^\alpha, \quad \alpha = m, \ m+2, \ m+4.$$

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