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ON THE EXISTENCE OF IDENTITIES IN IDEALS AND SUBRINGS

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Throughout the present paper, $R(\neq 0)$ will represent a ring with Jacobson radical $J=J(R)$. Recently, the second author and I. Murase [10, Theorem 1] and Dinh Van Huynh [2, Theorem 1] gave some necessary and sufficient conditions for the existence of a right (resp. left) identity in R with J nilpotent such that R/J has an identity. Furthermore, Dinh Van Huynh [2] investigated the conditions for the existence of a right identity in an ideal of a right Artinian ring.

In the present paper, we improve considerably the results in [2] (§2), extend [7, Lemma], [4, Proposition 4.4], [9, Theorem 3], [5, Theorem 1] and [3, Proposition 2.1] to subrings in place of ideals, and deduce some results in [8] (§3).

1. Let M be a non-zero left (resp. right) R-module. Following [9], M is called *s-unital* if $u\in Ru$ (resp. $u\in uR$) for all $u\in M$. If ${}_RR$ (resp. R_R) is s-unital, we term R a *left* (resp. *right*) *s-unital ring*. Given a left (resp. right) ideal I of R and a subset L of ${}_RM$ (resp. M_R), we set $I^{-1}L=\{u\in M|Iu\subseteq L\}$ (resp. $LI^{-1}=\{u\in M|uI\subseteq L\}$) and $r_M(I)\,(=I^{-1}0\,)=\{u\in M|Iu=0\}$ (resp. $l_M(I)=\{u\in M|uI=0\}$). Let $\Phi_r=\Phi_r(R)$ be the intersection of all maximal right ideals of R, and $K_r=K_r(R)=\{x\in R|xR\subseteq\Phi_r\}$. Then K_r is an ideal of R which is known as the (right) Kertész radical of R. As usual, the left (resp. right) annihilator of a subset S of R will be denoted by $l(S)$ (resp. $r(S)$).

In preparation for the subsequent study, we state the following lemmas.

Lemma 1. *Let M be a non-zero left R-module. Then the following are equivalent:*

1) *${}_RM$ is s-unital.*
2) *For every proper R-submodule M' of M, ${}_RM/M'$ is unital.*
3) *For every R-submodule M' of M, $R^{-1}M'=M'$.*
4) *For every finite subset U of M, there exists an element e in R such that $eu=u$ for all $u\in U$.*

Proof. See [9, Theorem 1].

Lemma 2. *Let I be a left ideal of a ring R. Then the following are equivalent*:

1) *I is left T-nilpotent.*
2) *For every non-zero right R-module M, $l_M(I) \neq 0$.*
3) *For every right R-module M, $l_M(I)$ is essential in M_R.*
4) *For every non-zero left R-module M, $IM \neq M$.*
5) *For every left R-module M, IM is small in ${}_RM$.*

Proof. The proof proceeds in the same way as that of [1, Proposition 8]. Only minor modifications are needed in proving 2)$\Rightarrow$4) and 5)$\Rightarrow$1).

Corollary 1. *Let I be a (left and right) T-nilpotent ideal of a ring R. If R/I has an identity and $RI = IR$, then R is a direct sum of a ring with identity and a T-nilpotent ring.*

Proof. Choose an idempotent e such that $e+I$ is the identity of R/I. Then

$$(R,+) = R_{11} \oplus R_{10} \oplus R_{01} \oplus R_{00},$$

where $R_{11} = eRe$, $R_{10} = eR(1-e)$, $R_{01} = (1-e)Re$, and $R_{00} = (1-e)R(1-e)$. Since $R_{10}+R_{01}+R_{00} \subseteq I$, we have

$$(I,+) = eIe \oplus R_{10} \oplus R_{01} \oplus R_{00}.$$

Now, it is easy to see that

$$(RI,+) = eIe \oplus R_{10} \oplus (R_{01}Ie+R_{00}R_{01}) \oplus (R_{01}R_{10}+R_{00}^2),$$
$$(IR,+) = eIe \oplus (eIR_{10}+R_{10}R_{00}) \oplus R_{01} \oplus (R_{01}R_{10}+R_{00}^2).$$

Since $RI = IR$, from these it follows that $R_{10} = eIR_{10}+R_{10}R_{00}$ and $R_{01} = R_{01}Ie+R_{00}R_{01}$. Since eIe is a left T-nilpotent ideal of R_{11}, $eIR_{10} = eIeR_{10}$ is small in R_{10} as left R_{11}-module (Lemma 2), and therefore $R_{10} = R_{10}R_{00}$. Similarly, $R_{10}R_{00}$ is small in R_{10} as right R_{00}-module, and consequently $R_{10} = 0$. A similar argument shows that $R_{01} = 0$, and therefore $R = R_{11} \oplus R_{00}$.

Let I be an ideal of R. An idempotent e of R is called a *right* (resp. *left*) *identity of R modulo I* if $e+I$ is a right (resp. left) identity of R/I.

Lemma 3. *Let I be a quasi-regular ideal of R. If R has a right identity e modulo I, then the following are equivalent*:

1) *R has a right identity.*

2) *J is a small left ideal of R.*
3) *I is a small left ideal of R.*

Proof. 1)$\Rightarrow$ 2). Since R is a cyclic left R-module, every proper left ideal of R is contained in some (modular) maximal left ideal. Hence, J is a small left ideal of R.

2)$\Rightarrow$ 3). Trivial.

3)$\Rightarrow$ 1). Since $R = Re + I$, we get $R = Re$ by 3).

Lemma 4. *Suppose $K_r = 0$. If R has a left identity e modulo J, then e is a right identity of R. In particular, if R has a left identity then R has an identity.*

Proof. Let a be an arbitrary element of R. Then $(a-ae)R = (a-ae)(eR+J) = (a-ae)J$. Since RJ is seen to be contained in all maximal right ideals of R, the last implies that $a-ae \in K_r$, namely $a-ae=0$.

2. In this section, we improve [2, Theorems 1, 3 and 7]. Our main theorem (Theorem 1) will be followed by corollaries which generalize [2, Theorem 10 and Corollary 11]. We begin with reproving [5, Lemma 1].

Proposition 1. (1) *A ring R has a left identity if and only if R is left s-unital and there exists an element c in R such that $l(c)$ is nil.*

(2) *A ring R has an identity if and only if R is left s-unital and there exists an element c in R such that $l(c)=0$.*

Proof. (1) Suppose that R is left s-unital and there exists $c \in R$ such that $l(c)$ is nil. Choose $b \in R$ with $bc = c$. Since $b^2 - b \in l(c)$, we have $(b^2-b)^m = 0$ for some positive integer m, and so $b^m b' - b^m = 0$ with some b' in the subring generated by b. Now, let a be an arbitrary element of R, and choose $d \in R$ such that $d(b'a-a) = b'a-a$. Since $d-db^m \in l(c)$, we have $(d-db^m)^k = 0$ with some positive integer k. On the other hand, noting that $b^m(b'a-a) = (b^m b' - b^m)a = 0$, we get $(d-db^m)(b'a-a) = b'a-a$. This together with $(d-db^m)^k = 0$ proves that $b'a-a=0$. The converse is trivial, since if e is a left identity of R then $(l(e))^2 = 0$.

(2) The existence of a right identity of R is rather familiar, so that R has an identity by (1).

We are now in a position to prove the following that improves [2, Theorem 1].

Theorem 1. *If I is a nil ideal of R and R/I has a right identity, then the following are equivalent*:

1) *R has a right identity.*
2) *J is a small left ideal of R.*
3) *I is a small left ideal of R.*
4) *R is right s-unital.*

If I is a left T-nilpotent ideal of R and R/I has a right identity, then 1)—4) *are equivalent to the following*:

5) *For every ideal H of R with $H\subseteq I$, $HR^{-1}=H$.*
6) *$IR=I$.*

Proof. In view of Lemma 3, it is enough to show that 4) implies 1). As is well known, R has a right identity e modulo I. Since $Rr(e)=(Re+I)r(e)\subseteq I$, $r(e)$ is a nil right ideal. Hence, by Proposition 1 (1), R has a right identity. We assume henceforth that I is left T-nilpotent, and show that 4)$\Rightarrow$5)$\Rightarrow$6)$\Rightarrow$3).

4)$\Rightarrow$5). Obvious by Lemma 1.

5)$\Rightarrow$6). Since $IR\subseteq I$, we have $(IR)R^{-1}=IR$. Combining this with the fact that $I\subseteq(IR)R^{-1}$, we readily obtain $I=IR$.

6)$\Rightarrow$3). By Lemma 2, IR is a small left ideal of R. Thus 6) implies 3).

Corollary 2. *If I is a T-nilpotent ideal of R and R/I has an identity, then the following are equivalent*:

1) *R has an identity.*
2) *I is a small left ideal and a small right ideal of R.*
3) *R is left and right s-unital.*
4) *$RI=IR=I$.*
5) *$RI=IR$ and $R^2=R$.*
6) *$RI=IR$ and $l(R)\cap r(R)=0$.*

Proof. The equivalence of 1)—4) is obvious by Theorem 1. It therefore remains to prove that each of 5) and 6) implies 1). Recall here that if 5) or 6) is satisfied then $R=U\oplus T$, where U has an identity and T is T-nilpotent (Corollary 1). If 5) is satisfied then the T-nilpotent ideal T is idempotent, and hence $T=0$ by Lemma 2. Henceforth, we assume that 6) is satisfied. Suppose. $T\neq 0$. In general, if $t_m\cdots t_1t_1'\cdots t'_{m'}\neq 0$ for some $t_1,\cdots,t_m,t_1',\cdots,t'_{m'}\in T$, then $l_T(T)\cap r_T(T)=0$ implies that $t_{m+1}t_m\cdots t_1t_1'\cdots t'_{m'}\neq 0$ for some $t_{m+1}\in T$ or $t_m\cdots t_1t_1'\cdots t'_{m'}t'_{m'+1}\neq 0$ for some $t'_{m'+1}\in T$. But, this contradicts the T-nilpotency of T. Thus, $T=0$, and so $R = U$.

The next includes [2, Theorem 10].

Corollary 3. *Suppose that R has a left identity e modulo J. If K_r is nil then the following are equivalent*:

1) *R has a right identity.*
2) *J is a small left ideal of R.*
3) *K_r is a small left ideal of R.*
4) *R is right s-unital.*

If K_r is left T-nilpotent then 1) — 4) *are equivalent to the following*:

5) *For every ideal H of R with $H \subseteq K_r$, $HR^{-1} = H$.*
6) *$K_rR = K_r$.*

Proof. It is easy to see that $K_r(R/K_r) = 0$. Since $e + K_r$ is a left identity of R/K_r modulo $J(R/K_r) = J/K_r$, $e + K_r$ is a right identity of R/K_r by Lemma 4. Thus, we can apply Theorem 1 for $I = K_r$.

As an easy combination of Theorem 1 and Corollary 3, we obtain the following generalization of [2, Corollary 11].

Corollary 4. *If J is left T-nilpotent and R/J has an identity, then the following are equivalent*:

1) *R has a right identity.*
2) *J is a small left ideal of R.*
3) *K_r is a small left ideal of R.*
4) *R is right s-unital.*
5) *For every ideal H of R with $H \subseteq J$, $HR^{-1} = H$.*
6) *For every ideal H of R with $H \subseteq K_r$, $HR^{-1} = H$.*
7) *$JR = J$.*
8) *$K_rR = K_r$.*

In the rest of this section, our interest will be exclusively directed toward the existence of identities in ideals.

A ring R is called a *left* (resp. *right*) *perfect ring* if J is left (resp. right) T-nilpotent and R/J is Artinian. (In case R has an identity, it is well known that R is a right perfect ring if and only if R satisfies the minimum condition for principal left ideals.)

Now, [2, Theorems 3 and 7] can be improved as follows:

Theorem 2. *Let A be an ideal of R.*

(1) *If J is nil and R/J is Artinian then the following are equivalent*:

1) *A has a right identity.*

2) *$J(A)$ is a small left ideal of A.*

3) *A is right s-unital.*

If R is left perfect then 1)—3) *are equivalent to the following*:

4) *For every ideal H of A with $H \subseteq J(A)$, $HA^{-1} = H$ (in A).*

5) *$J(A)A = J(A)$.*

(2) *If R is a (left and right) perfect ring, then the following are equivalent*:

1) *A has an identity.*

2) *$J(A)$ is a small left ideal and a small right ideal of A.*

3) *A is s-unital.*

4) *$AJ(A) = J(A)A = J(A)$.*

5) *$AJ(A) = J(A)A$ and $A^2 = A$.*

6) *$AJ(A) = J(A)A$ and $l_A(A) \cap r_A(A) = 0$.*

Proof. In either case, $J(A) = A \cap J$ and $A/J(A)$ is isomorphic to the semisimple Artinian ring $(A+J)/J$. Hence, (1) and (2) are immediate by Theorem 1 and Corollary 2, respectively.

Finally, we note that [2, Corollary 4] can be improved as follows:

Proposition 2. *Let A be a left ideal of a left π-regular ring R. Then A has a right identity if (and only if) there exists $c \in A$ such that $l_A(c) = 0$.*

Proof. By hypothesis, there exists a positive integer n and an element e in A such that $c^n = ec^n$. Then $c = ec$ and e is a right identity of A.

3. First, we generalize the key lemma of [7] as follows:

Proposition 3. *Let R be a ring satisfying the maximum condition for left annihilators. If A is a subring of R, then the following are equivalent*:

1) *A has a left identity.*

2) *A contains an element a such that $aA = A$ and $bA = A$* for all *$b \in A$ with $ab = a$.*

Proof. It is enough to show that 2) implies 1). According to 2), there exists an infinite sequence $\{a_n\}$ of elements in A such that $a_{n-1}a_n = a_{n-1}$ $(a_0 = a)$ and $a_nA = A$. Then $r(a_{n-1}) \supseteq r(a_n)$ for all n. Since R satisfies the minimum condition for right annihilators, there is a positive integer k such that $r(a_{k-1}) = r(a_k)$. Then, as can be easily seen, $aa_k = a$.

Now, given $x \in A$, there exists $y \in A$ such that $x = a_k y$. Since $a_{k-1}(y - a_k y) = 0$, we see that $0 = a_k(y - a_k y) = x - a_k x$, namely a_k is a left identity of A.

Corollary 5. *Let I be an ideal of a right Artinian ring R. Let a be an element of R. If $a^* = a + I$ is an idempotent in $R^* = R/I$, then aR contains an idempotent e such that $e^* = a^*$.*

Proof. Let A be a right ideal of R which is minimal with respect to the property that $A \subseteq aR$ and $A^* = a^* R^*$. Without loss of generality, we may assume that a is in A. Then, since $a^* A^* = a^{*2} R^* = a^* R^*$ and $aA \subseteq A$, we obtain $aA = A$. Next, if b is an element in A such that $ab = a$, then $b^* A^* = a^* b^* A^* = a^* A^* = a^* R^*$ and $bA \subseteq A$, and therefore $bA = A$. Thus, A contains a left identity e with $ae = a$, by Proposition 3 and its proof. It is now immediate that $e^* = a^* e^* = a^*$.

We are now in a position to state the first main theorem of this section that generalizes [4, Proposition 4.4], [5, Theorem 1] and [9, Theorem 3].

Theorem 3. *Let A be a subring of a left Goldie ring R. Then the following are equivalent:*

1) *A has a left identity.*
2) *A is left s-unital.*
3) *A contains an element a such that $aA = A$ and $bA = A$ for all $b \in A$ with $ab = a$.*

Proof. The equivalence of 1) and 3) is included in Proposition 3. Thus, in view of Proposition 1 (1), it suffices to show that there exists an element of A whose left annihilator in A is nil. Suppose on the contrary that A contains no element whose left annihilator in A is nil. We shall show that there exists an infinite sequence $\{a_n\}$ of non-nilpotent elements of A such that $a_n \in l_A(a_1 + \cdots + a_{n-1})$ and $(Ra_1 + \cdots + Ra_n) \cap l(a_1 + \cdots + a_n) = 0$, which will complete the proof. Choose first a non-nilpotent element a_1 in A such that $l(a_1)$ is maximal in the (non-empty) family $\{l(a) \mid a \in A$ is not nilpotent$\}$. Observe $Ra_1 \cap l(a_1) = 0$. Now, we proceed by induction. Let $b_n = a_1 + \cdots + a_n$, and choose a non-nilpotent element a_{n+1} in $l_A(b_n)$ such that $l(a_{n+1})$ is maximal in the family $\{l(a) \mid a \in l_A(b_n)$ is not nilpotent$\}$. Observe that $Ra_{n+1} \cap l(a_{n+1}) = 0$. Since $(Ra_1 + \cdots + Ra_n) \cap l(b_n) = 0$, we can easily see that $l(b_n + a_{n+1}) =$

$l(b_n) \cap l(a_{n+1})$. Then

$$\begin{aligned}(Ra_1+\cdots+Ra_{n+1}) \cap l(b_n+a_{n+1}) \\ &= [\{(Ra_1+\cdots+Ra_n) \cap l(b_n)\}+Ra_{n+1}] \cap l(a_{n+1}) \\ &= Ra_{n+1} \cap l(a_{n+1}) = 0.\end{aligned}$$

This completes the induction.

Following [6], we say that a ring R is *almost left Artinian* (resp. *almost left Noetherian*) if for each infinite descending (resp. ascending) chain $I_1 \supseteq I_2 \supseteq \cdots$ (resp. $I_1 \subseteq I_2 \subseteq \cdots$) of left ideals of R there exists a positive integer k such that $R^k I_k \subseteq I_n$ (resp. $R^k I_n \subseteq I_k$) for all n.

Corollary 6. (1) *Let R be an almost left Noetherian ring, and A a subring of R. If A is left s-unital then A has a left identity.*

(2) *Let R be an almost left Artinian ring, and A a subring of R. If A is left s-unital then A has a left identity.*

Proof. (1) As was claimed in the proof of [6, Theorem 3], there exists a positive integer q such that $R/r(R^q)$ is left Goldie. Noting that A is left s-unital, we can easily see that $A \cap r(R^q)=0$. Hence, A is isomorphic to the subring $(A+r(R^q))/r(R^q)$ of the left Goldie ring $R/r(R^q)$, and consequently A has a left identity by Theorem 3.

(2) This is obvious by (1) and [6, Theorem 4].

The next generalizes [3, Proposition 2.1] and [2, Theorem 14].

Theorem 4. *Let R be a ring satisfying the maximum condition for right annihilators. If A is a left s-unital subring of R, then A has a left identity.*

Proof. Given $a \in A$, we set $I(a)=r(R(1-a))$. By hypothesis, we can select an element e in A such that $I(e)$ is maximal in the family of all such right annihilators $I(a)$ $(a \in A)$. Suppose $A \nsubseteq I(e)$, and choose $b \in A$ such that $b \notin I(e)$. By Lemma 1, there exists an element e' in A such that $e'e=e$ and $e'b=b$. Then b is obviously in $I(e')$. Moreover, if x is in $I(e)$ then $(e'-e)x=e'(1-e)x=0$, and so $R(1-e')x=R((1-e)x-(e'-e)x)=0$. Thus we have $I(e) \subsetneq I(e')$. This contradiction shows that $A \subseteq I(e)$, whence if follows that $A(a-ea)=0$ for all $a \in A$. This implies that e is a left identity of A.

Combining Theorem 4 with Corollary 6 (1), we readily obtain

Corollary 7. *Let R be a one-sided Noetherian ring, and A a subring of R. If A is left (resp. right) s-unital then A has a left (resp. right) identity.*

As an application of Corollary 7, we prove [8, Proposition 3.11] (see also [4, Corollary 2.3]).

Corollary 8. *If the polynomial ring $R[X]$ is right Noetherian, then R has a right identity.*

Proof. It is a routine to show that R is right s-unital. Hence, R has a right identity by Corollary 7.

The next is a partial extension of [2, Theorem 3] in another direction.

Corollary 9. *Let R be a one-sided Artinian ring, and A a subring of R. If A is left s-unital then A has a left identity.*

Proof. Obviously, $R^2 \supseteq A^2 = A$. Since R^2 is left or right Noetherian according as R is left or right Artinian (see [10, Theorem 4 (c)]), A has a left identity by Corollary 7.

We end by generalizing [8, Corollary 2.4] as follows:

Corollary 10. *Let A be a subring of a one-sided Noetherian (resp. Artinian) ring R. If every right ideal of A is a right annihilator and every left ideal of A is a left annihilator, then A has an identity.*

Proof. It is immediate that A is an s-unital ring. Hence, A has an identity by Corollary 7 (resp. Corollary 9).

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