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ON ISOMORPHISM INVARIANTS OF INTEGRAL GROUP RINGS

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Let G be a group, $\boldsymbol{Z}G$ denote its integral group ring. Let $\{\gamma_i(G)\}$, $\{\delta_i(G)\}$ and $\{D_i(G)\}$ denote the lower central series, derived series and integral dimension series of G, respectively. The series $\{\gamma_i(G)\}$ and $\{\delta_i(G)\}$ are defined inductively by $\gamma_1(G)=G$, $\delta_1(G)=G$ and for $i \geqq 1$

$$\gamma_{i+1}(G)=(G,\gamma_i(G)),\ \delta_{i+1}(G)=(\delta_i(G),\delta_i(G)).$$

As usual, for any two subgroups A and B of G, (A,B) is the subgroup generated by all commutators $(a,b)=a^{-1}b^{-1}ab$, $a\in A$, $b\in B$. Denoting by $\Delta(G)$ the augmentation ideal of $\boldsymbol{Z}G$, the series $\{D_i(G)\}$ is defined by $D_i(G)=G\cap(1+\Delta(G)^i)$. We note here that for every group G, $D_2(G)=\gamma_2(G)=\delta_2(G)$ and $D_3(G)=\gamma_3(G)$ (see e.g. [4]).

In this note we find the subquotients of G which are isomorphism invariants of $\boldsymbol{Z}G$. It is known [6, p.100, Theorem 4.28] that if G is locally finite, then the successive subquotients $\gamma_i(G)/\gamma_{i+1}(G)$ and $\delta_i(G)/\delta_{i+1}(G)$ are isomorphism invariants of $\boldsymbol{Z}G$ for all $i\geqq 1$. Also, it is known [6, p.102, Theorem 5.3] that if G is torsion, then the quotient group $G/\delta_3(G)$ is an isomorphism invariant of $\boldsymbol{Z}G$. Extending these results we prove the following two theorems.

Theorem A. *Suppose $\boldsymbol{Z}G\cong \boldsymbol{Z}H$ where G is a torsion group. Then, for all $i\geqq 1$, we have the following isomorphisms*:

(i) $\gamma_i(G)/\gamma_{i+j}(G)\cong\gamma_i(H)/\gamma_{i+j}(H)$ *and* $D_i(G)/D_{i+j}(G)\cong D_i(H)/D_{i+j}(H)$ *for all j with $1\leqq j\leqq i$,*

(ii) $\delta_i(G)/\delta_{i+1}(G)\cong\delta_i(H)/\delta_{i+1}(H)$.

Theorem B. *Suppose $\boldsymbol{Z}G\cong\boldsymbol{Z}H$. If $G/\gamma_2(G)$ is torsion, then*

$$G/\delta_3(G)\cong H/\delta_3(H),\ G/\gamma_3(G)\cong H/\gamma_3(H),\ G/\gamma_4(G)\cong H/\gamma_4(H).$$

Moreover if G is torsion, then $G/D_4(G)\cong H/D_4(H)$.

In what follows, unless otherwise stated, G will denote an arbitrary group. Also, for a normal subgroup N of G we denote by $\Delta(G,N)$ the kernel of the natural homomorphism from $\boldsymbol{Z}G$ to $\boldsymbol{Z}(G/N)$ and by $V(\boldsymbol{Z}G)$ the units of augmentation one of the integral group ring $\boldsymbol{Z}G$.

In order to prove our results we need some well-known facts which are collected below (see e.g. [3, 5 and 6]).

(I) Let N be a normal subgroup of G. Then

$$N/(G,N) \cong \Delta(G,N)/\Delta(G)\Delta(G,N)+\Delta(G,N)\Delta(G), \text{ and}$$
$$(G,N) = G \cap (1+\Delta(G)\Delta(G,N)+\Delta(G,N)\Delta(G)).$$

(II) Let N be a normal subgroup of G. Then

$$N/(N,N) \cong \Delta(G,N)/\Delta(G)\Delta(G,N), \text{ and}$$
$$(N,N) = G \cap (1+\Delta(G)\Delta(G,N)).$$

(III) Let N and L be normal subgroups of G such that $(N,N) \subseteqq L \subseteqq N$. Then

$$N/L \cong \Delta(G,N)/\Delta(G)\Delta(G,N)+\Delta(G,L), \text{ and}$$
$$L = G \cap (1+\Delta(G)\Delta(G,N)+\Delta(G,L)).$$

(IV) Let $u = \sum_g u(g)g \in V(\boldsymbol{Z}G)$ be of finite order with $u(1) \neq 0$. Then $u = 1$.

(V) If G is torsion abelian then $TV(\boldsymbol{Z}G) = G$, where $TV(\boldsymbol{Z}G)$ is the torsion subgroup of $V(\boldsymbol{Z}G)$.

1. Proof of Theorem A. Using (IV) the same argument as in Lemma 3.1 of [1] gives us the following result, whose proof will be omitted.

Lemma 1.1. *If X is a periodic subgroup of $V(\boldsymbol{Z}G)$, then the elements of X are linearly independent over $\boldsymbol{Z}$, the ring of rational integers.*

A series $G = N_1 \supseteqq N_2 \supseteqq \cdots \supseteqq N_n \supseteqq \cdots$ of subgroups of G is called an N-series of G if $(N_n,N_m) \subseteqq N_{n+m}$ for all $n,m \geqq 1$. The familiar examples of N-series of G are its lower central series $\{\gamma_i(G)\}$ and integral dimension series $\{D_i(G)\}$.

Lemma 1.2. *Let $\{N_i\}$ be an N-series of G. Then*

$$N_i/N_{i+j} \cong \Delta(G,N_i)/\Delta(G)\Delta(G,N_i)+\Delta(G,N_{i+j})$$

for all $i \geqq 1$ and all j with $1 \leqq j \leqq i$.

Proof. Since $(N_i,N_i) \subseteqq N_{2i} \subseteqq N_{i+j}$, the result is an immediate consequence of (III).

We recall that a (ring) homomorphism f from $\boldsymbol{Z}G$ to the integral group ring $\boldsymbol{Z}H$ is called *normalized* if the sum of all the coefficients of $f(g)$ is 1 for all $g \in G$.

Lemma 1.3. *Let G be a torsion group. Then any normalized epimorphism $f : \boldsymbol{Z}G \rightarrow \boldsymbol{Z}H$ satisfying $G \cap (1+Ker\ f) = \{1\}$ is an isomorphism.*

Proof. It is easy to see that the restriction of f to G is an imbedding into $V(\boldsymbol{Z}H)$. Hence the result follows from Lemma 1.1.

We now adapt the method of Cohn-Livingstone [1] to prove Theorem A. Suppose that we have a normalized isomorphism $\theta : \boldsymbol{Z}G \rightarrow \boldsymbol{Z}H$. Then, following [1], for any normal subgroup N of G the normal subgroup ϕN of H is defined by $\phi N = H \cap (1+\theta(\Delta(G,N)))$. (Note that we do not identify $\boldsymbol{Z}H$ with $\boldsymbol{Z}G$.) Similarly, for any normal subgroup K of H we define $\phi^* K = G \cap (1+\theta^{-1}(\Delta(H,K)))$. From the definition of ϕ we readily see that $\theta(\Delta(G,N)) \supseteq \Delta(H,\phi N)$. Furthermore we have

Lemma 1.4. (1) *If N is a normal subgroup of G such that $H/\phi N$ is torsion, then $\theta(\Delta(G,N)) = \Delta(H,\phi N)$.*

(2) *Let I be an ideal of $\boldsymbol{Z}G$ and set $N = G \cap (1+I)$, $K = H \cap (1+\theta(I))$. If $G/\phi^* K$ is torsion, then $\phi N = K$.*

Proof. (1) Since $\theta(\Delta(G,N)) \supseteq \Delta(H,\phi N)$, there is a natural epimorphism $\boldsymbol{Z}(H/\phi N) \rightarrow \boldsymbol{Z}H/\theta(\Delta(G,N))$, which induces a normalized epimorphism $f : \boldsymbol{Z}(H/\phi N) \rightarrow \boldsymbol{Z}(G/N)$ such that $H/\phi N \cap (1+Ker\ f) = \{1\}$. Hence by Lemma 1.3, f is an isomorphism, which implies that $\theta(\Delta(G,N)) = \Delta(H,\phi N)$.

(2) Since $\Delta(G,N) \subseteq I$ and $\Delta(H,K) \subseteq \theta(I)$, it follows that $\phi N \subseteq K$ and $\phi^* K \subseteq N$. On the other hand, since $G/\phi^* K$ is torsion, applying (1) of this lemma to θ^{-1} and ϕ^* we get $\theta^{-1}(\Delta(H,K)) = \Delta(G,\phi^* K)$. This means that $K = \phi\phi^* K$ and $K \subseteq \phi N$. Thus $\phi N = K$.

1.5. Proof of Theorem A. Let $\theta : \boldsymbol{Z}G \rightarrow \boldsymbol{Z}H$ be the given isomorphism; then we may assume here that θ is normalized. Let ϕ be defined as above. In view of (II) and Lemma 1.2, to complete the proof of the theorem it remains only to prove that for all $i \geqq 1$

$$\theta(\Delta(G,\gamma_i(G))) = \Delta(H,\gamma_i(H)), \tag{1.6}$$

$$\theta(\Delta(G,D_i(G))) = \Delta(H,D_i(H)), \tag{1.7}$$

$$\theta(\Delta(G,\delta_i(G))) = \Delta(H,\delta_i(H)). \tag{1.8}$$

First, we use induction on i to see (1.6), the case $i=1$ being obvious. Assume that for some $i \geqq 1$, we have

$$\theta(\Delta(G,\gamma_k(G))) = \Delta(H,\gamma_k(H)) \text{ for all } k \text{ with } 1 \leqq k \leqq i.$$

Then it follows by (I) that $\gamma_k(G)/\gamma_{k+1}(G) \cong \gamma_k(H)/\gamma_{k+1}(H)$ for all k with $1 \leqq k \leqq i$. From this we deduce that $H/\gamma_{i+1}(H)$ is torsion. Noting here that

$$\gamma_{i+1}(G) = G \cap (1+\Delta(G)\Delta(G,\gamma_i(G))+\Delta(G,\gamma_i(G))\Delta(G))$$

by (I), it follows from Lemma 1.4 (2) and the induction hypothesis that

$$\begin{aligned}\phi\gamma_{i+1}(G) &= H \cap (1+\theta\{\Delta(G)\Delta(G,\gamma_i(G))+\Delta(G,\gamma_i(G))\Delta(G)\}) \\ &= H \cap (1+\Delta(H)\Delta(H,\gamma_i(H))+\Delta(H,\gamma_i(H))\Delta(H)) \\ &= \gamma_{i+1}(H)\end{aligned}$$

and so, $H/\phi\gamma_{i+1}(G)$ is torsion. Hence by Lemma 1.4 (1), we have

$$\theta(\Delta(G,\gamma_{i+1}(G))) = \Delta(H,\phi\gamma_{i+1}(G)) = \Delta(H,\gamma_{i+1}(H))$$

which completes the induction on i. Thus (1.6) is established. Next we shall prove (1.7). By Lemma 1.4 (2), we obtain

$$\phi D_i(G) = H \cap (1+\theta(\Delta(G)^i)) = H \cap (1+\Delta(H)^i) = D_i(H)$$

for all $i \geqq 1$. Since each $H/\gamma_i(H)$ is torsion as seen by using (1.6), so is each $H/\phi D_i(G)$ because $\gamma_i(H) \subseteqq D_i(H) = \phi D_i(G)$. Hence by Lemma 1.4 (1), $\theta(\Delta(G,D_i(G))) = \Delta(H,D_i(H))$ proving (1.7). Finally, the proof of (1.8) being similar to that of (1.6) (using (II)), we omit the details.

2. Proof of Theorem B. Given an ideal I of ZG, we mean by $G+I$ the subset (of ZG) of all elements of the form $g+\alpha$, $g \in G$, $\alpha \in I$. The following lemma can be easily verified.

Lemma 2.1 (see [2, Lemma 1]). *Let $\theta : ZG \to ZH$ be an isomorphism. Let I be an ideal of ZG such that $\theta(G+I) = H+\theta(I)$. Then we have the following isomorphism:*

$$G/G \cap (1+I) \cong H/H \cap (1+\theta(I)).$$

We recall that for any normal subgroup N of G, $\Delta(G,N)$ is the ideal of ZG generated by all $x-1$, $x \in N$.

Lemma 2.2. *Let N and L be normal subgroups of G, with $N \supseteqq L$. Then, $\Delta(G)\Delta(G,N)+\Delta(G,L)\Delta(G) = \Delta(G)\Delta(G,N)+\Delta(G,(G,L))$.*

Proof. The equality follows easily from the identity $(l-1)(g-1) = (g-1)(l-1) - lg((g,l)-1)$ for $l \in L$ and $g \in G$.

We now define the ideals $I(G)$, $I_3(G)$, $I_4(G)$ and $J_4(G)$ of $\boldsymbol{Z}G$ by setting

$$I(G) = \Delta(G)\Delta(G,\gamma_2(G)),\ I_3(G) = I(G)+\Delta(G,\gamma_3(G)),$$
$$I_4(G) = I(G)+\Delta(G,\gamma_4(G)) \text{ and } J_4(G) = I(G)+\Delta(G,D_4(G)).$$

Using Lemma 2.2 we have

$$I(G)+\Delta(G,\gamma_2(G))\Delta(G) = I(G)+\Delta(G,\gamma_3(G)) = I_3(G),$$
$$I(G)+\Delta(G,\gamma_3(G))\Delta(G) = I(G)+\Delta(G,\gamma_4(G)) = I_4(G),$$

and moreover

$$I(G)+\Delta(G,\gamma_3(G))\Delta(G) = I(G)+\{I(G)+\Delta(G,\gamma_3(G))\}\Delta(G) =$$
$$I(G)+\{I(G)+\Delta(G,\gamma_2(G))\Delta(G)\}\,\Delta(G) = I(G)+\Delta(G,\gamma_2(G))\,\Delta(G)^2.$$

Therefore, we observe that

(2.3) $I_3(G) = I(G)+\Delta(G,\gamma_2(G))\Delta(G)$ and $I_4(G) = I(G)+\Delta(G,\gamma_2(G))\Delta(G)^2$.

Also, from (II) and (III) we obtain:

(2.4)
$$G \cap (1+I(G)) = \delta_3(G),\ G \cap (1+I_3(G)) = \gamma_3(G),$$
$$G \cap (1+I_4(G)) = \gamma_4(G) \text{ and } G \cap (1+J_4(G)) = D_4(G).$$

We are now in a position to prove Theorem B.

2.5. Proof of Theorem B. We may assume that the given isomorphism $\theta : \boldsymbol{Z}G \to \boldsymbol{Z}H$ is normalized, and therefore $\theta(\Delta(G)) = \Delta(H)$. Since $\Delta(G,\gamma_2(G))$ is the smallest ideal I such that $\boldsymbol{Z}G/I$ is commutative, it follows that $\theta(\Delta(G,\gamma_2(G))) = \Delta(H,\gamma_2(H))$, and hence $\theta(I(G)) = I(H)$. Further, we deduce by (2.3) that $\theta(I_3(G)) = I_3(H)$ and $\theta(I_4(G)) = I_4(H)$. Now, suppose that $G/\gamma_2(G)$ is torsion. Then, also $H/\gamma_2(H)$ is torsion, since we have $G/\gamma_2(G) \cong H/\gamma_2(H)$. We claim

(2.6) $$\theta(G+I(G)) = H+I(H).$$

As a matter of fact, the proof of (2.6) can be found in [7]. However, for the sake of completeness, we give its proof. Let $g \in G$, and let $\pi : \boldsymbol{Z}H \to \boldsymbol{Z}(H/\gamma_2(H))$ be the natural homomorphism. Then $\pi(\theta(g))$ is a torsion element of $V(\boldsymbol{Z}(H/\gamma_2(H)))$ and thus by (V), $\pi(\theta(g)) = \pi(h_1)$ for some $h_1 \in H$, so $\theta(g) = h_1 + \alpha$ where α is in $\Delta(H,\gamma_2(H))$. By (II), $\gamma_2(H)/\delta_3(H) \cong \Delta(H,\gamma_2(H))/I(H)$ and, this isomorphism being given by $h\delta_3(H) \to h-1+I(H)$, there exists $h_2 \in \gamma_2(H)$ such that $\alpha \equiv h_2 - 1 \pmod{I(H)}$.

So, $\theta(g) \equiv h_1+h_2-1 = h_1h_2-(h_1-1)(h_2-1) \equiv h_1h_2 \pmod{I(H)}$, which implies that $\theta(g) \in H+I(H)$. We have therefore proved that $\theta(G) \subseteqq H+I(H)$. As seen above, $\theta(I(G)) = I(H)$, so $\theta(G+I(G)) \subseteqq H+I(H)$. The same argument shows that $\theta^{-1}(H+I(H)) \subseteqq G+I(G)$, and hence (2.6) has been proved. It is clear from (2.6) that $\theta(G+I_3(G)) = H+I_3(H)$ and $\theta(G+I_4(G)) = H+I_4(H)$. Hence by Lemma 2.1 and (2.4) we conclude that

$$G/\delta_3(G) \cong H/\delta_3(H),\ G/\gamma_3(G) \cong H/\gamma_3(H),\ G/\gamma_4(G) \cong H/\gamma_4(H).$$

Finally, if G is torsion then $\theta(\Delta(G,D_4(G))) = \Delta(H,D_4(H))$ by (1.7), so that $\theta(J_4(G)) = J_4(H)$. Therefore, we have $\theta(G+J_4(G)) = H+J_4(H)$ and hence $G/D_4(G) \cong H/D_4(H)$, which completes the proof of the theorem.

Corrigendum. In Theorem in [2] the inequalities $1 \leqq i \leqq j+1$ are false and should be replaced by $1 \leqq j \leqq i+1$.

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