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A CLASSIFICATION OF ABELIAN EXTENSIONS OF RINGS

Dedicated to Prof. Kentaro Murata on his 60th birthday

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Introduction. Let B be a ring with identity element 1. In [4], one of the present authors has studied on some classifications of free cyclic extensions of B of ρ-automorphism type and of D-derivation type.

In this paper, as a natural sequel of [4], we shall prove similar results for free abelian extensions. For this purpose, we need notions of some skew polynomial rings of several variables which will be seen later on. §1 is devoted to a classification of free (abelian) extensions of P-automorphism type of B and §2 is devoted to that of free abelian extensions of D-derivation type of an algebra B over GF(p).

Throughout Z denotes the center of B. For any subring S of B, $U(S)$ means the set of inversible elements in S. Moreover, for any $u \in U(B)$ (resp. $b \in B$), $\tilde{u}$ (resp. I_b) means the inner automorphism $u_l u_r^{-1}$ (resp. derivation $b_r - b_l$) effected by u (resp. b).

Now, let (P, E) (resp. (D, F)) be a pair of a finite set $P = \{\rho_1, \cdots, \rho_e\}$ (resp. $D = \{D_1, \cdots, D_e\}$) of automorphisms (resp. derivations) of B and a subset $E = \{b_{i,j}\,;\, i, j = 1, \cdots, e\}$ (resp. $F = \{c_{i,j}\,;\, i, j = 1, \cdots, e\}$) of B satisfying the following conditions (i), (ii) and (iii) (resp. (i)', (ii)' and (iii)'):

(i) $b_{ij}b_{ji} = 1$ and $b_{ii} = 1$

(ii) $\rho_i \rho_j \rho_i^{-1} \rho_j^{-1} = \tilde{b}_{ij}$

(iii) $b_{ij}\rho_j(b_{ik})b_{jk} = \rho_i(b_{jk})b_{ik}\rho_k(b_{ij})$

for all $i, j, k = 1, \cdots, e$.

(i)' $c_{ij} + c_{ji} = 0$ and $c_{ii} = 0$

(ii)' $[D_i, D_j] = D_iD_j - D_jD_i = I_{c_{ji}}$

(iii)' $D_i(c_{jk}) + D_k(c_{ij}) + D_j(c_{ki}) = 0$

for all $i, j, k = 1, \cdots, e$.

Then, as is observed in [2] and [3], the free right B-module with a B-basis $\{X_1^{\nu_1}X_2^{\nu_2}\cdots X_e^{\nu_e};\ \nu_i \geq 0\}$ becomes an associative ring by the rule (a) (resp. (a)')

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(a) $bX_i = X_i\rho_i(b),\ X_iX_j = X_jX_ib_{ij} \qquad (b\in B,\ 1\le i,\ j\le e)$

(a)′ $bX_i = X_ib + D_i(b),\ X_iX_j = X_jX_i + c_{ij} \quad (b\in B,\ 1\le i,\ j\le e)$

This is called a skew polynomial ring of (P, E)-*automorphism type* (abbr. P-*automorphism type*) (resp. (D, F)-*derivation type* (abbr. D-*derivation type*)), which will be denoted by $B[X;P,E]$ (resp. $B[X;D,F]$).

Moreover, we use the following conventions:

$B^{\rho_i} = \{b\in B;\ \rho_i(b) = b\},\ B^P = \cap_{i=1}^e B^{\rho_i},\ Z^P = Z\cap B^P,$

$B^{D_i} = \{b\in B;\ D_i(b) = 0\},\ B^D = \cap_{i=1}^e B^{D_i},\ Z^D = Z\cap B^D,$

S^e = the direct product of e-copies of a subset of S of B. Further, for $\mathfrak{a} = (a_i),\ \mathfrak{b} = (b_i)\in B^e$ and $\mathfrak{c} = (c_i)\in U(B)^e$, as usual, we write

$$\mathfrak{a}\mathfrak{b} = (a_ib_i),\ \mathfrak{a}+\mathfrak{b} = (a_i+b_i) \text{ and } \mathfrak{c}^{-1} = (c_i^{-1}).$$

1. A classification of free extensions of P-automorphism type. In this section, G means an abelian group $(\sigma_1)\times\cdots\times(\sigma_e)$ with $|\sigma_i| = n_i$ and $n = \prod_{i=1}^e n_i$. Moreover, we assume the following

(1) $U(Z)$ contains a primitive n-th root ζ of 1, $1-\zeta^i$ $(1\le i\le n-1)$ and n.

(2) There exists a pair (P,E) of a finite set $P = \{\rho_1, \cdots, \rho_e\}$ of automorphisms of B and a subset $E = \{b_{ij};\ 1\le i, j\le e\}$ of B satisfying the conditions (i), (ii), (iii) and $\rho_i(\zeta) = \zeta$ for all $i = 1, \cdots, e$.

Now, for convenience, we set

$A = B[X;P,E],\ N_i(b) = b\rho_i(b)\rho_i^2(b)\cdots\rho_i^{n_i-1}(b)\ (b\in B),$

$B_i = \{b\in B^{\rho_i};\ cb = b\rho_i^{n_i}(c)$ for all $c\in B$ and $b = \rho_j(b)N_i(b_{ji})$ for $j = 1, \cdots, e\}$ $(b_{ji}\in E)$,

$\mathfrak{B} = B_1\times B_2\times\cdots\times B_e$ (direct product),

$M(\mathfrak{b}) = \sum_{i=1}^e (X_i^{n_i} - b_i)A$ for $\mathfrak{b} = (b_1, \cdots, b_e)\in\mathfrak{B}$; then $M(\mathfrak{b})$ is a two-sided ideal of A, and we write

$A(\mathfrak{b})$ = the factor ring of A modulo $M(\mathfrak{b})$.

Let $\mathfrak{b}\in\mathfrak{B}$, and $x_i = X_i + M(\mathfrak{b})$ $(\in A(\mathfrak{b}))$. Then $\{x_1^{\mu_1}\cdots x_e^{\mu_e};\ 0\le\mu_i<n_i\}$ is a right free B-basis of $A(\mathfrak{b})$ which satisfies $x_i^{n_i} = b_i,\ x_ix_j = x_jx_ib_{ij}$ and $cx_i = x_i\rho_i(c)$ for every $c\in B$. If we define the action of G on $A(\mathfrak{b})$ by $\sigma_i(x_jc) = x_j\zeta_j^\varepsilon c$ where $\zeta_j = \zeta^{n/n_j}$ and $\varepsilon = \delta_{ij}$, then G can be considered as a B-automorphism group of $A(\mathfrak{b})$ and if $\mathfrak{b}\in\mathfrak{B}\cap U(B)^e$ then $A(\mathfrak{b})$ becomes a strongly G-Galois extension of B (see [3]). For $\mathfrak{b}\in\mathfrak{B}$, (resp. $\mathfrak{b}\in\mathfrak{B}\cap U(B)^e$), $A(\mathfrak{b})$ will be called a *free* (resp. *free abelian*) extension of B of P-automorphism type. Moreover, we set

$\Omega = \Omega(B;P,E) = \{A(\mathfrak{b});\ \mathfrak{b}\in\mathfrak{B}\}$,

$\langle A(\mathfrak{b})\rangle = \{A(\mathfrak{c})\in\Omega;\ A(\mathfrak{c})$ is BG-ring isomorphic to $A(\mathfrak{b})\}$,

$\Lambda = \Lambda(B;P,E) =$ the set of all the classes $\langle A(\mathfrak{b})\rangle$ $(\mathfrak{b} \in \mathfrak{B})$,
$U = U(B;P,E) = \{\langle A(\mathfrak{b})\rangle;\ \mathfrak{b} \in \mathfrak{B} \cap U(B)^e\}$.

As is proved in [3], if R is a strongly G-abelian extension of B with $B_B \langle\oplus R_B$, then R is obtained by $A(\mathfrak{b}')$ $(\mathfrak{b}' \in U(B)^e)$ for some (P',E') which satisfies the condition (2). Hence U means the set of all BG-ring isomorphism classes of strongly G-abelian extensions R of B which are of (P,E)-automorphism type with $B_B \langle\oplus R_B$.

In the rest of this section, we assume that U is non vacuous. Further, by $\mathfrak{u} = (u_1, \cdots, u_e)$, we denote some element of $\mathfrak{B} \cap U(B)^e$. Then $\rho_i^{n_i} = \tilde{u}_i$ and we have the following.

Lemma 1.1. *$B_i = u_i Z^P$, and $B_i \cap U(B) = u_i U(Z^P)$ $(1 \le i \le e)$.*

Proof. As is easily seen, we have

$$u_i Z \cap \{b \in B;\ b = \rho_j(b) N_i(b_{ji}),\ j = 1, \cdots, e\} = u_i Z^P$$

for all $i = 1, \cdots, e$. Combining this with [3, Lemma 1.2], we obtain the assertion.

Lemma 1.2. *Let $A(\mathfrak{b})$ be in Ω, and $G_i = (\sigma_{i+1}) \times \cdots \times (\sigma_e)$ $(0 \le i < e)$. Then $A(\mathfrak{b})^{G_i} = B[x_1, \cdots, x_i]$, where $x_i = X_i + M(\mathfrak{b})$.*

Proof. Obviously $A(\mathfrak{b})^{\sigma_e} \supset B[x_1, \cdots, x_{e-1}]$. Conversely, let $\sum_{k=0}^{n_e-1} x_e^k f_k \in A(\mathfrak{b})^{\sigma_e}$ where $f_k = B[x_1, \cdots, x_{e-1}]$. Noting $\sigma_e(x_e) = x_e \zeta_e$ and $1 - \zeta_e^k \in U(B)$ $(1 \le k \le n_e - 1)$, we have $f_k = 0$ $(1 \le k \le n_e - 1)$. Hence $A(\mathfrak{b})^{\sigma_e} = B[x_1, \cdots, x_{e-1}]$. By induction method, we obtain the assertion.

Now, we set
$\mathfrak{Z} = \{\mathfrak{r} = (r_1, \cdots, r_e) \in U(Z)^e;\ \rho_j(r_k) r_j = \rho_k(r_j) r_k,\ 1 \le j, k \le e\}$,
$N(\mathfrak{r}) = (N_1(r_1), \cdots, N_e(r_e))$ for any $\mathfrak{r} = (r_i) \in \mathfrak{Z}$,
$N(\mathfrak{Z}) = \{N(\mathfrak{r});\ \mathfrak{r} \in \mathfrak{Z}\}$.

Let $\mathfrak{r} \in \mathfrak{Z}$. Then, by the condition (ii) of (2), we have $\rho_i \rho_j(r_k) = \rho_j \rho_i(r_k)$ $(1 \le i, j, k \le e)$. Hence $\rho_j(N_k(r_k)) = N_k(\rho_j(r_k)) = N_k(\rho_j(r_k) r_j r_j^{-1}) = N_k(\rho_k(r_j) r_k r_j^{-1}) = N_k(r_k)$ $(j = 1, \cdots, e)$. Thus we obtain $N(\mathfrak{r}) \in U(Z^P)^e$. It is easy to see that $N(\mathfrak{Z})$ is a subgroup of $U(Z^P)^e$ Next, we shall prove the following

Lemma 1.3. *Let $A(\mathfrak{b})$ and $A(\mathfrak{c})$ be in Ω. Set $x_i = X_i + M(\mathfrak{b})$ and $y_i = X_i + M(\mathfrak{c})$ $(i = 1, \cdots, e)$. Then the following conditions are equivalent.*

(1) $\langle A(\mathfrak{b})\rangle = \langle A(\mathfrak{c})\rangle$.

(2) *There exists a B-ring isomorphism ϕ of $A(\mathfrak{b})$ to $A(\mathfrak{c})$ such that* $\phi(x_i) = y_i r_i$ *with* $r_i \in U(Z)$ $(i = 1, \cdots, e)$.

(3) $\mathfrak{b} = \mathfrak{c}N(\mathfrak{r})$ *for some* $\mathfrak{r} \in \mathfrak{Z}$.

Proof. Assume (1). Let ϕ be a BG-ring isomorphism of $A(\mathfrak{b})$ to $A(\mathfrak{c})$, and $H_i = (\sigma_1)\times\cdots\times(\sigma_{i-1})\times(\sigma_{i+1})\times\cdots\times(\sigma_e)$. Then $A(\mathfrak{b})^{H_i} = B[x_i]$ by Lemma 1.2. Since $\phi\tau = \tau\phi$ for every $\tau \in H_i$, we have $\phi(B[x_i]) \subset A(\mathfrak{c})^{H_i} = B[y_i]$. Considering ϕ^{-1}, we obtain $\phi(B[x_i]) = B[y_i]$. Hence, it follows from [4, Lemma 1.1. (2)] that $\phi(x_i) = y_i r_i$ for some $r_i \in U(Z)$. This proves (1) $\Rightarrow$ (2).

Assume (2). Obviously $b(y_j r_j) = (y_j r_j)\rho_j(b)$ for all $b \in B$ $(j = 1, \cdots, e)$. Since $x_j x_k = x_k x_j b_{jk}$ $(j, k = 1, \cdots, e)$, we have $(y_j r_j)(y_k r_k) = (y_k r_k)\cdot(y_j r_j) b_{jk}$. This implies that $\rho_k(r_j) r_k = \rho_j(r_k) r_j$ $(j, k = 1, \cdots, e)$. Moreover, we have $b_i = \phi(x_i^{n_i}) = (y_i r_i)^{n_i} = y_i^{n_i} N_i(r_i) = c_i N_i(r_i)$ $(i = 1, \cdots, e)$. Thus we obtain (3).

Assume (3). Let Φ be the map of A to A defined by

$$\sum X_1^{\nu_1} X_2^{\nu_2} \cdots X_e^{\nu_e} b_\nu \to \sum (X_1 r_1)^{\nu_1} (X_2 r_2)^{\nu_2} \cdots (X_e r_e)^{\nu_e} b_\nu.$$

Obviously $b(X_j r_j) = (X_j r_j)\rho_j(b)$ for all $b \in B$. Noting $\rho_k(r_j) r_k = \rho_j(r_k) r_j$ $(1 \leq j, k \leq e)$, we have $(X_j r_j)(X_k r_k) = (X_k r_k)(X_j r_j) b_{jk}$ $(1 \leq j, k \leq e)$. Hence, it follows that Φ is a B-ring isomorphism. Since $\Phi(X_i^{n_i} - b_i) = (X_i r_i)^{n_i} - b_i = X_i^{n_i} N_i(r_i) - c_i N_i(r_i) = (X_i^{n_i} - c_i) N_i(r_i)$, we have $\Phi(M(\mathfrak{b})) = M(\mathfrak{c})$. Thus Φ induces a B-ring isomorphism Φ of $A(\mathfrak{b})$ to $A(\mathfrak{c})$ and it is easy to see that Φ is BG-linear. This completes the proof.

Now, we are in a position to porove the following

Theorem 1.4. (1) *Λ forms an abelian semigroup under the composition* $\langle A(\mathfrak{b})\rangle * \langle A(\mathfrak{c})\rangle = \langle A(\mathfrak{u}^{-1}\mathfrak{b}\mathfrak{c})\rangle$ *with the identity* $\langle A(\mathfrak{u})\rangle$.

(2) $U(\Lambda) = U$ *and is isomorphic to the factor group* $U(Z^P)^e / N(\mathfrak{Z})$.

Proof. (1). If $\langle A(\mathfrak{b})\rangle = \langle A(\mathfrak{b}')\rangle$ and $\langle A(\mathfrak{c})\rangle = \langle A(\mathfrak{c}')\rangle$, then there are $\mathfrak{r}$ and $\mathfrak{s} \in \mathfrak{Z}$ such that $\mathfrak{b} = \mathfrak{b}' N(\mathfrak{r})$ and $\mathfrak{c} = \mathfrak{c}' N(\mathfrak{s})$ (Lemma 1.3). Then $\mathfrak{u}^{-1}\mathfrak{b}\mathfrak{c} = \mathfrak{u}^{-1}\mathfrak{b}' N(\mathfrak{r})\mathfrak{c}' N(\mathfrak{s}) = \mathfrak{u}^{-1}\mathfrak{b}'\mathfrak{c}' N(\mathfrak{r}\mathfrak{s})$, and $\mathfrak{u}^{-1}\mathfrak{b}\mathfrak{c} \in \mathfrak{B}$ (Lemma 1.1). Thus the composition $*$ is well defined. The other assertion can be easily seen.

(2). It is clear that $U(\Lambda) = \{\langle A(\mathfrak{b})\rangle ; \mathfrak{b} \in \mathfrak{B} \cap U(B)^e\} = U$. Let f be the map of $U(Z^P)^e$ to U defined by $f(\mathfrak{v}) = \langle A(\mathfrak{u}\mathfrak{v})\rangle$. Then f is an epimorphism by Lemma 1.1. If $f(\mathfrak{v}) = \langle A(\mathfrak{u})\rangle$, then there is $\mathfrak{r} \in Z$ such that $\mathfrak{v} = N(\mathfrak{r})$. The converse is also true. Hence $U(\Lambda) \simeq U(Z^P)^e / \ker f \simeq U(Z^P)^e / N(\mathfrak{Z})$.

Now, the pair $(P|Z, \{1\} = \{b_{ij} = 1 (1 \leq i, j \leq e\})$ satisfies the condition (2) over Z. Hence $A(\mathfrak{v}) = Z[X; P|Z, \{1\}]/M(\mathfrak{v})$ with $\mathfrak{v} \in (Z^P)^e$ is a free extension of Z of $(P|Z)$-automorphism type and $\Lambda(Z; P|Z, \{1\})$ forms an abelian semigroup with the identity $\langle A(1)\rangle$ by $\langle A(\mathfrak{v})\rangle * \langle A(\mathfrak{w})\rangle = (A(\mathfrak{v}\mathfrak{w}))$. Considering the map $\Lambda(Z; P|Z, \{1\}) \to \Lambda$ defined by $\langle A(\mathfrak{v})\rangle \to \langle A(\mathfrak{u}\mathfrak{v})\rangle$, we obtain the following

Corollary 1.5. *Λ is isomorphic to $\Lambda(Z; P|Z, \{1\})$.*

As a direct consequence of Theorem 1.4, we obtain the following

Corollary 1.6. *Assume that $\rho_i|Z = 1$ for $i = 1, \cdots, e$. Then $U \simeq \prod_i U(Z)/U(Z)^{n_i}$ (direct product). In particular, if B is commutative and $\rho_i = 1$ for $i = 1, \cdots, e$ then $U \simeq \prod_i U(B)/U(B)^{n_i}$.*

Remark. (1) By Theorem 1.4, the group structure of $U(\Lambda)$ does not depend on the choice of $\mathfrak{u}$.

(2) If (P, E') satisfies the condition (2) and $U(B; P, E')$ is non vacuous then $\Lambda(B; P, E')$ is isomorphic to Λ by Corollary 1.5. Thus, if we assume that $\rho_i\rho_j = \rho_j\rho_i$ (and hence $\tilde{b}_{ij} = 1$) for $i, j = 1, \cdots, e$ and $U(B; P, \{1\})$ is non vacuous, then Λ is isomorphic to $\Lambda(B; P, \{1\})$.

2. A classification of free extensions of D-derivation type. In this section, G means an elementary abelian group $(\sigma_1) \times \cdots \times (\sigma_e)$ of order p^e for a prime p. Moreover, we assume the following

(1)′ B is an algebra over GF(p).

(2)′ There exists a pair (D, F) of a finite set $D = \{D_1, \cdots, D_e\}$ of derivations of B and a subset $F = \{c_{ij}; 1 \leq i, j \leq e\}$ of B satisfying the conditions (i)′, (ii)′ and (iii)′.

Now, we set

$A = B[X; D, F]$,

$B_i = \{b \in B^{D_i}; I_b = D_i^p - D_i, (D_i^{p-1} - 1)(c_{ji}) + D_j(b) = 0$ for $j = 1, \cdots, e\}$ $(c_{ji} \in F)$,

$\mathfrak{B} = B_1 \times B_2 \times \cdots \times B_e$ (direct product),

$M(\mathfrak{b}) = \sum_{i=1}^{e}(X_i^p - X_i - b)A$ for $\mathfrak{b} = (b_i) \in \mathfrak{B}$; then $M(\mathfrak{b})$ is a two-sided ideal of A, and we write

$A(\mathfrak{b}) =$ the factor ring of A modulo $M(\mathfrak{b})$.

Let $\mathfrak{b} \in \mathfrak{B}$ and $x_i = X_i + M(\mathfrak{b}) (\in A(\mathfrak{b}))$. Then $\{x_1^{\mu_1} \cdots x_e^{\mu_e}; 0 \leq \mu_i \leq p-1\}$ is a right free B-basis for $A(\mathfrak{b})$ which satisfies $x_i^p - x_i = b_i$, $x_i x_j = x_j x_i + c_{ij}$

and $cx_i = x_i c + D_i(c)$ for every $c \in B$. If we define the action of G on $A(\mathfrak{b})$ by $\sigma_i(x_j c) = (x_j + \delta_{ij})c$, then G can be considered as a B-automorphism group of $A(\mathfrak{b})$ and $A(\mathfrak{b})$ becomes a G-abelian extension of B with $B_B \langle\oplus\, A(\mathfrak{b})_B$ (see [2]). Moreover, we set

$\Omega = \Omega(B;D,F) = \{A(\mathfrak{b});\ \mathfrak{b} \in B\}$,
$\langle A(\mathfrak{b})\rangle = \{A(\mathfrak{c}) \in \Omega;\ A(\mathfrak{c})$ is BG-ring isomorphic to $A(\mathfrak{b})\}$,
$\Lambda = \Lambda(B;D,F) =$ the set of all the classes $\langle A(\mathfrak{b})\rangle$ $(\mathfrak{b} \in \mathfrak{B})$.

As is proved in [2], if R is a G-abelian extension of B with $B_B \langle\oplus\, R_B$, then R is obtained by $A(\mathfrak{b}')$ for some (D',F') which satisfies the condition (2)'. Hence Λ means the set of all BG-ring isomorphism classes of G-abelian extensions R of B which are of (D,F)-derivation type with $B_B \langle\oplus\, R_B$.

In the rest of this section, we assume that $\mathfrak{B}$ is non vacuous. Further, by $\mathfrak{u} = (u_1, \cdots, u_e)$, we denote some element of $\mathfrak{B}$. Then $D_i^p - D_i = I_{u_i}$ and we have the following which is corresponding to Lemma 1.1.

Lemma 2.1. $B_i = u_i + Z^D$ $(1 \leq i \leq e)$.

Proof. As is easily seen, we have

$$(u_i + Z) \cap \{b \in B;\ (D_i^{p-1} - 1)(c_{ji}) + D_j(b) = 0,\ j = 1, \cdots, e\} = u_i + Z^D.$$

Combining this with [4, Lemma 2.3], we obtain the assertion.

Lemma 2.2. *Let $A(\mathfrak{b})$ be in Ω, and $G_i = (\sigma_{i+1}) \times \cdots \times (\sigma_e)(0 \leq i < e)$. Then $A(\mathfrak{b})^{G_i} = B[x_1, \cdots, x_i]$, where $x_i = X_i + M(\mathfrak{b})$.*

Proof. In virtue of the results of [2], we see that $A(\mathfrak{b})$ is a G_i-abelian extension of $B[x_1, \cdots, x_i]$, which implies the assertion.

Now, we set

$\mathfrak{Z} = \{\mathfrak{r} = (r_1, \cdots, r_e) \in Z^e;\ D_i(r_j) = D_j(r_i),\ 1 \leq i, j \leq e\}$,
$d_i(r) = (D_i^{p-1} - 1)(r) + r^p$ for any $r \in Z$,
$d(\mathfrak{r}) = (d_1(r_1), \cdots, d_e(r_e))$ for $\mathfrak{r} = (r_i) \in \mathfrak{Z}$,
$d(\mathfrak{Z}) = \{d(\mathfrak{r});\ \mathfrak{r} \in \mathfrak{Z}\}$.

Then, by the same arguments as in that of [1, p. 190], we can prove that $d_i(r) = (X_i + r)^p - X_i^p - r$ for any $r \in Z$. Moreover, d_i is an additive homomorphism of Z to itself. Hence $d(\mathfrak{Z})$ is a subgroup of $(Z^D)^e$.

Next, we shall prove the following

Lemma 2.3. *Let $A(\mathfrak{b})$ and $A(\mathfrak{c})$ be in Ω. Let $x_i = X_i + M(\mathfrak{b})$ and*

$y_i = X_i + M(\mathfrak{c})$ $(i = 1, \cdots, e)$. *Then the following conditions are equivalent.*

(1) $\langle A(\mathfrak{b})\rangle = \langle A(\mathfrak{c})\rangle$.

(2) *There exists a B-ring isomorphism* Φ *of* $A(\mathfrak{b})$ *to* $A(\mathfrak{c})$ *such that* $\Phi(x_i) = y_i + r_i$ *with* $r_i \in Z$.

(3) $\mathfrak{b} = \mathfrak{c} + d(\mathfrak{r})$ *for some* $\mathfrak{r} \in \mathfrak{Z}$.

Proof. Assume (1). Let Φ be a BG-ring isomorphism of $A(\mathfrak{b})$ to $A(\mathfrak{c})$. Then, noting the result of Lemma 2.2, we can prove $\Phi(B[x_i]) = B[y_i]$ $(1 \leq i \leq e)$ by making use of the same methods as in the proof of (1)$\Rightarrow$(2) in Lemma 1.3. Hence, it follows from [4, Lemma 2.2 (2)] that $\Phi(x_i) = y_i + r_i$ for some $r_i \in Z$.

Assume (2). Obviously $b(y_i + r_i) = (y_i + r_i)b + D_i(b)$ for all $b \in B$ $(i = 1, \cdots, e)$. Since $x_j x_k = x_k x_j + c_{jk}$ $(j, k = 1, \cdots, e)$, we have $(y_j + r_j)\cdot(y_k + r_k) = (y_k + r_k)(y_j + r_j) + c_{jk}$. This means that $D_k(r_j) = D_j(r_k)$ $(j, k = 1, \cdots, e)$. Moreover, we have $b_i = \Phi(x_i^p - x_i) = (y_i + r_i)^p - (y_i + r_i) = c_i + d_i(r_i)$ $(i = 1, \cdots, e)$. Thus we obtain (3).

Assume (3). Let Φ be the map of A to A defined by

$$\sum X_1^{\nu_1} \cdots X_e^{\nu_e} b_\nu \rightarrow \sum (X_1 + r_1)^{\nu_1} \cdots (X_e + r_e)^{\nu_e} b_\nu.$$

Noting that $b(X_j + r_j) = (X_j + r_j)b + D_j(b)$ and $D_k(r_j) = D_j(r_k)$ $(b \in B$, $1 \leq j,k \leq e)$, we have $(X_j + r_j)(X_k + r_k) = (X_k + r_k)(X_j + r_j) + c_{jk}$ $(1 \leq j, k \leq e)$. Hence, it follows that Φ is a B-ring isomorphism. Since $\Phi(X_i^p - X_i - b_i) = (X_i + r_i)^p - (X_i + r_i) - b_i = X_i^p - X_i + d_i(r_i) - b_i = X_i^p - X_i - c_i$, we have $\Phi(M((\mathfrak{b}))) = M(\mathfrak{c})$. Thus Φ induces a B-ring isomorphism ϕ of $A(\mathfrak{b})$ to $A(\mathfrak{c})$ and it is easy to see that ϕ is BG-linear. This completes the proof.

Now, we shall prove the following which is corresponding to Theorem 1.4.

Theorem 2.4. (1) Λ *forms an abelian group under the composition* $\langle A(\mathfrak{b})\rangle * \langle A(\mathfrak{c})\rangle = \langle A(\mathfrak{b} + \mathfrak{c} - \mathfrak{u})\rangle$ *with the dinetity* $\langle A(\mathfrak{u})\rangle$.

(2) Λ *is isomorphic to the factor group* $(Z^D)^e / d(\mathfrak{Z})$.

Proof. (1). If $\langle A(\mathfrak{b})\rangle = \langle A(\mathfrak{b}')\rangle$ and $\langle A(\mathfrak{c})\rangle = \langle A(\mathfrak{c}')\rangle$ then there are $\mathfrak{r}$ and $\mathfrak{s} \in \mathfrak{Z}$ such that $\mathfrak{b} = \mathfrak{b}' + d(\mathfrak{r})$ and $\mathfrak{c} = \mathfrak{c}' + d(\mathfrak{s})$ (Lemma 2.3). Then $\mathfrak{b} + \mathfrak{c} - \mathfrak{u} = \mathfrak{b}' + \mathfrak{c}' - \mathfrak{u} + d(\mathfrak{r} - \mathfrak{s})$. Thus the composition $*$ is well defined. The other assertions can be easily seen.

(2). Let f be the map of $(Z^D)^e \rightarrow \Lambda$ defined by $f(\mathfrak{v}) = \langle A(\mathfrak{v} + \mathfrak{u})\rangle$. Then f is an epimorphism by Lemma 2.1, and the kernel is $d(\mathfrak{Z})$ by Lemma 2.3. Hence $\Lambda \simeq (Z^D)^e / d(\mathfrak{Z})$.

Now, the pair $(D|Z, \{0\} = \{c_{ij} = 0\ (1 \leq i, j \leq e)\})$ satisfies the condition $(2)'$ over Z. Hence $A(\mathfrak{v}) = Z[X;D|Z,\{0\}]/M(\mathfrak{v})$ with $\mathfrak{v} \in (Z^D)^e$ is a free extension of Z of $(D|Z)$-derivation type and $\Lambda(Z;D|Z,\{0\})$ forms an abelian group with the identity $\langle A(0)\rangle$ by $\langle A(\mathfrak{v})\rangle * \langle A(\mathfrak{w})\rangle = \langle A(\mathfrak{v}+\mathfrak{w})\rangle$. Considering the map $\Lambda(Z;D|Z,\{0\}) \rightarrow \Lambda$ defined by $\langle A(\mathfrak{v})\rangle \rightarrow \langle A(\mathfrak{u}+\mathfrak{v})\rangle$, we obtain

Corollary 2.5. *Λ is isomorphic to $\Lambda(Z;D|Z,\{0\})$.*

If $D_i = 0$ then $d_i(r) = r^p - r$ for any $r \in Z$. Hence the following is a direct consequence of Theorem 2.4.

Corollary 2.6. *Assum $D_i|Z = 0$ for $i = 1, \cdots, e$. Then $\Lambda \simeq (Z/Z^{\mathfrak{p}})^e$ where $Z^{\mathfrak{p}} = \{r^p - r;\ r \in Z\}$. In particular, if B is commutative and $D_i = 0$ for $i = 1, \cdots, e$ then $\Lambda \simeq (B/B^{\mathfrak{p}})^e$.*

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