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DOUBLE SUSPENSIONS OF BRIESKORN INVOLUTIONS

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The purpose of this paper is to give an explicit description of double suspensions of the Brieskorn involutions of dimensions 5 and 13.

For each odd positive integer d, let Σ_d^{4k+1} be the Brieskorn sphere which is the submanifold of C^{2k+2} described by the equations

$$z_0^d+z_1^2+\cdots+z_{2k+1}^2=0 \text{ and } z_0\bar{z}_0+z_1\bar{z}_1+\cdots+z_{2k+1}\bar{z}_{2k+1}=1.$$

Then Σ_d^{4k+1} is a homotopy sphere and the involution $T_d:\Sigma_d^{4k+1}\to\Sigma_d^{4k+1}$ given by $T_d(z_0)=z_0$ and $T_d(z_i)=-z_i(i>0)$ is a fixed point free involution of Σ_d^{4k+1}. The involution (T_d, Σ_d^{4k+1}) is obtained also from the equivariant plumbing of $(d-1)$ copies of the tangent disk bundle of S^{2k+1} around the fixed points.

In § 1, we apply an equivariant plumbing technique to construct homotopy projective spaces of dimensions 7 and 15, and obtain the following: For each odd positive integer d, there exists a free involution T_d on a homotopy sphere Σ_d^7 (resp. Σ_d^{15}) such that (1) T_d has the Brieskorn involution (T_d, Σ_d^5) (resp. (T_d, Σ_d^{13})) as a desuspension, (2) $\Sigma_d^7\in bP_8$ (resp. $\Sigma_d^{15}\in bP_{16}$), (3) T_d extends to an involution with isolated fixed points on a 3-connected (resp. 7-connected) 8-dimensional (resp. 16-dimensional) spin manifold M_d^8 (resp. M_d^{16}) which is bounded by Σ_d^7 (resp. Σ_d^{15}), and (4) $a(T_d, \Sigma_f^7)=\pm d \bmod 2^4$ (resp. $a(T_d, \Sigma_d^{15})=\pm d \bmod 2^8$) (Theorem 1. 1 and 1. 2). Next, in § 2, we correct the results on plumbing manifolds stated in [5] as follows: $28\mu(\Sigma_{2h+1}^7)=[(h+1)/2] \bmod 28$ and $2^6\cdot 127\mu(\Sigma_{2h+1}^{15})=[(h+1)/2] \bmod 2^6\cdot 127$ (Theorem 2. 1).

In § 3, we modify the Bredon construction [3] to give other examples of homotopy projective 15-spaces with the 13-dimensional Brieskorn involutions as desuspensions. By means of the Milnor-Munkres-Novikov pairing, Bredon has obtained some exotic actions on some elements of θ^{n+k} from the linear action on the sphere S^{n+k}. This construction will be generalized for free involutions on homotopy spheres. Applying it to the examples in Theorem 1. 2, we see that for each odd positive integer d, there exists a free involution $(T'_d, \Sigma_d'^{15})$ with $\Sigma_d'^{15}\notin bP_{16}$ which has the Brieskorn involution (T_d, Σ_d^{13}) as a desuspension (Theorem 3. 7). Moreover, we determine the double suspensions of the Brieskorn involution (T_d, Σ_d^5) and (T_d, Σ_d^{13}): The double suspension of the Brieskorn involution (T_d, Σ_d^5) is

(T_d, Σ_d^7), modulo the action of bP_8 (Proposition 3.1), and the double suspensions of the Brieskorn involution (T_d, Σ_d^{13}) are exactly Σ_d^{15}/T_d, $\Sigma_d^{15}/T_d \# \Sigma_s^{15}$, $\Sigma_d'^{15}/T'_d$, $\Sigma_d'^{15}/T'_d \# \Sigma_s^{15}$, modulo the action of bP_{16}, where Σ_s^{15} is the generator on the Z_2-summand in θ^{15} which bounds no parallelizable manifolds (Theorem 3.11).

In § 4, we first improve the estimate for the spin invariants of the Brieskorn involutions given in [8, p. 338] (Proposition 4.1), and give an answer to the question raised in [8, p. 337] concerning the classification of $hS(P^5)$, the set of homotopy smoothings of the standard projective 5-space P^5 (Corollary 4.2). Secondly, we correct a result of Yang [12] concerning the determination of the desuspensions of Hirsch-Milnor involutions (Proposition 4.3). Finally, we claim that the nonexistence of double suspensions of the Brieskorn involutions is equivalent to the Kervaire invariant conjecture as follows: For $k=2^r-1$, $r>1$, there is a free involution T on a $(4k+3)$-dimensional homotopy sphere Σ^{4k+3} which has the Brieskorn involution (T_d, Σ_d^{4k+1}) as a desuspension for $d=\pm 3 \bmod 8$ if and only if Σ_d^{4k+1} is diffeomorphic to the standard sphere S^{4k+1}.

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1. Construction of certain homotopy projective spaces. We give some examples of homotopy projective spaces of dimensions 7 and 15.

Theorem 1.1. *For each odd positive integer d, there is a free involution T_d on a homotopy sphere Σ_d^7 with the following properties,*

(1) *T_d has the Brieskorn involution (T_d, Σ_d^5) as a desuspension.*

(2) *$\Sigma_d^7 \in bP_8$, where bP_8 is the group of homotopy spheres which bound parallelizable manifolds.*

(3) *T_d extends to an involution with isolated fixed points on a 3-connected 8-dimensional spin manifold M_d^8 which is bounded by Σ_d^7. The spin invariant $a(T_d, \Sigma_d^7)$ of (T_d, Σ_d^7) is $\pm d \bmod 2^4$.*

Theorem 1.2. *For each odd positive integer d, there is a free involution T_d on a homotopy sphere Σ_d^{15} with the following properties,*

(1) *T_d has the Brieskorn involution (T_d, Σ_d^{13}) as a desuspension.*

(2) *$\Sigma_d^{15} \in bP_{16}$, where bP_{16} is the group of homotopy spheres which bound parallelizable manifolds.*

(3) *T_d extends to an involution with isolated fixed points on a 7-connected 16-dimensional spin manifold M_d^{16} which is bounded by Σ_d^{15}. The spin invariant $a(T_d, \Sigma_d^{15})$ is $\pm d \bmod 2^8$.*

Proof of Theorem 1. 2. Let D^8 be the unit disk in the Cayley numbers $\mathfrak{C}$ with the Z_2-action, $t(x_0, x_1, \cdots, x_7)=(-x_0, -x_1, \cdots, -x_7)$. Let S^7 be the boundary ∂D^8 of D^8. Let S^8 be the suspension of S^7, i. e., the unit sphere in $\mathfrak{C}\times R$ with the Z_2-action, $t(x_0, x_1, \cdots x_7, y)=(-x_0, -x_1, \cdots, -x_7, y)$. Let $f_{h,j}: S^7 \longrightarrow SO(8)$ be the map defined by $f_{h,j}(u)(v)=(u^h v)u^j$ for $u\in S^7$, $v\in D^8$. Fix $f=f_{1,-1}$. Then f is invariant under the above Z_2-action on S^7. So f factors through a map $g: P^7 \longrightarrow SO(8)$,

$$\begin{array}{ccc} S^7 & \xrightarrow{\ f\ } & SO(8) \\ {\scriptstyle\pi}\downarrow & \nearrow_{g} & \\ P^7 & & \end{array},$$

where π is the projection map. We define a D^8-bundle E over S^8 with a Z_2-action T by $E^{16}=(D^8\times D^8)\underset{b(g)}{\cup}(D^8\times D^8)$, where $b(g)$: $S^7\times D^8 \longrightarrow S^7\times D^8$, $b(g)(u, v)=(u^{-1}, g\pi(u)(v))$ and the Z_2-action is given by $T(u, v)=(-u, -v)$ for $(u, v)\in D^8\times D^8$.

This involution T has two isolated fixed points on the zero-section of E. The euler class $e(E)$ is zero (in general, if one forgets the action, then the map $f_{h,j}$ induces a bundle $\xi_{h,j}$ over S^8 with euler calss $e(\xi_{h,j})=\pm(h+j)\iota$, and Pontrjagin calss $P_2(\xi_{h,j})=\pm 6(h-j)\iota$, where ι is the cofundamental class of $H^8(S^8)$). We show that

(1) (T, E^{16}) has an invariant characteristic submanifold (T, E^{14}), where E^{14} is the tangent disk bundle $E(\tau_{S^7})$ of S^7. We let $D^7=\{u\in D^8, \mathrm{Re}(u)=0\}$ ($\mathrm{Re}(u)$ is the real part of the Cayley number u), and we denote by S^6 the unit sphere of D^7. Obviously $\mathrm{Re}(uvu^{-1})=\mathrm{Re}(v)=0$, whenever $u\in S^6$ and $v\in D^7$. Defining the map g_1: $P^6 \longrightarrow SO(7)$ by $g_1(\pi(u))(v)=uvu^{-1}$, we see that the following diagram is commutative:

$$\begin{array}{ccc} P^7 & \xrightarrow{\ g\ } & SO(8) \\ \cup & & \cup \\ P^6 & \xrightarrow{\ g_1\ } & SO(7) \end{array}.$$

Hence (T, E^{16}) has an invariant characteristic submanifold

$$(T, E^{14})=(D^7\times D^7)\underset{b(g_1)}{\cup}(D^7\times D^7)\subset(T, E^{16}).$$

Since $g_1\pi(u)(v)=-uvu$, $g_1\pi$ is just the characteristic map of the tangent bundle τ_{S^7} of S^7, hence it follows (1).

We next prove that the spin invariant $\alpha(T, \partial E^{16})$ is $\pm 2 \bmod 2^8$ (cf. [1], [2]). The manifolds E^{14}, E^{16} are spin manifolds since $H^j(E^{14})=H^j(E^{16})=0$ for $j=1, 2$. The inclusion $i: (T, E^{14}) \hookrightarrow (T, E^{16})$ has the trivial nornal bundle, so it induces an embedding $F(i): FE^{14} \longrightarrow FE^{16}$, where FE^k is the space of oriented orthonormal k-frames for $k=14, 16$. Then, from the homotopy exact sequence of the fibration $SO(k) \longrightarrow FE^k \longrightarrow E^k$, we have that $F(i)_*: \pi_1(FE^{14}) \longrightarrow \pi_1(FE^{16})$ is an isomorphism. Recalling that $(T, E^{14}) = (T, E(\tau_{S^7}))$ in (1), we see that $a(T, \partial E^{14}) = \pm 2 \bmod 2^7$, i.e., the two fixed points have the same sign. Then, the proposition 8.44[1] assures us that $a(T, \partial E^{16}) = \pm 2 \bmod 2^8$, since the image of the loop in FE^{14} by the map $F(i)$ is the loop in FE^{16}.

For each positive integer h, we plumb $2h$ copies of the bundles E^{16} equivariantly at the fixed point on each. Denote by (β_h, M_h^{16}) the resulting manifold with Z_2-action. Since the euler class of E^{16} is zero, the plumbing matrix is equal to

$$\left.\begin{pmatrix} 0 & 1 & & & & 0 \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 0 & 1 \\ 0 & & & & 1 & 0 \end{pmatrix}\right\} 2h$$

We can easily check that the determinant is ± 1, and hence ∂M_h^{16} is a homotopy sphere. We note that $\partial M_h^{16} \in bP_{16}$. This follows from the Lemma 7 [11]. Put

$$(\beta_h | \partial M_h^{16}, \partial M_h^{16}) = (T_{2h+1}, \Sigma_{2h+1}^{15}).$$

Then M_h^{16} contains $(2h+1)$ fixed points, and by the contribution of $a(T, \partial E^{16}) = \pm 2 \bmod 2^8$ we have

(2) $a(T_{2h+1}, \Sigma_{2h+1}^{15}) = \pm(2h+1) \bmod 2^8$.

On the other hand, we see by (1) that (β_h, M_h^{16}) has a codimension 2 invariant characteristic submanifold which is just the resulting of the equivariant plumbing of $2h$ copies of $E^{14} = E(\tau_{S^7})$. If the resulting manifold is denoted by (β_h, M_h^{14}), then as was noted above, we have $(\beta_h | \partial M_h^{14}, \partial M_h^{14}) = (T_{2h+1}, \Sigma_{2h+1}^{13})$ (the Brieskorn involution). Furthermore, we recall that the desuspension invariant for $(\beta_h | \partial M_h^{16}, \partial M_h^{16})$ is the index of the above plumbing matrix, which is clearly zero. Let $(T_{2h+1}, \Sigma_{2h+1}^{14})$ be a desuspension of $(T_{2h+1}, \Sigma_{2h+1}^{15})$. Since free involutions of even dimensional homotopy spheres always desuspend, and their invariant characteristic spheres are equivariantly diffeomorphic, $(T_{2h+1}, \Sigma_{2h+1}^{13})$ is a desuspension of $(T_{2h+1}, \Sigma_{2h+1}^{14})$. This completes the proof of Theorem 1.2.

Proof of Theorem 1.1 is similar to that of Theorem 1.2 by using the quaternion field instead of the Cayley numbers, so we omit it.

2. Differentiable structures of Σ_d^{4k+3}. Applying the Eells-Kuiper μ-invariant [5], we are able to compute the differentiable structures of Σ_{2h+1}^7 $(=\partial M_h^8)$ and Σ_{2h+1}^{15} $(=\partial M_h^{16})$. (Unfortunately, the results on plumbing manifolds in [5] are incorrect since the contribution $(j^*)^{-1}$ is missing.)

Theorem 2. 1. *The differentiable structures of* Σ_{2h+1}^7, Σ_{2h+1}^{15} *are given by*

(1) $28\mu(\Sigma_{2h+1}^7)=[(h+1)/2] \bmod 28$,

(2) $2^6\cdot 127\mu(\Sigma_{2h+1}^{15})=[(h+1)/2] \bmod 2^6\cdot 127$,

where 28 (*resp.* $2^6\cdot 127$) *are the order of* bP_8 (*resp.* bP_{16}), *and* $[*]$ *is the integral part of* $*$.

Proof. Let A_h be the plumbing matrix of rank $2h$ introduced in the proceeding section. We consider the following commutative diagram,

$$\begin{array}{ccc} H_4(M_h^8) & \xrightarrow{j_*} & H_4(M_h^8, \partial M_h^8) \\ \downarrow D & & \downarrow D \\ H^4(M_h^8, \partial M_h^8) & \xrightarrow{j^*} & H^4(M_h^8). \end{array}$$

j_* is represented by A_h with respect to the standard basis of $H_4(M_h^8)$ and $H_4(M_h^8, \partial M_h^8)$. Let $u_i\in H^4(M_h^8)$ $(i=1, \cdots, 2h)$ be the basis corresponding to the standard basis of $H_4(M_h^8, \partial M_h^8)$ by the Poincaré duality isomorphism D. The i-th block of the Pontrjagin class $P_1(M_h^8)$ is $P_1(E^8)=P_1(\xi_{1,-1})=-2^2u_i$. Hence by the construction of M_h^8, it follows that $P_1(M_h^8)=-\sum_{i=1}^{2h}2^2u_i$. Since the index on $H_4(M_h^8)$ is equal to that of A_h, which is clearly zero, by the definition of μ ([5]) we have $28\mu(\Sigma_{2h+1}^7)=(j^{*-1}P_1(M_h^8))^2/2^5$. Now let B_h be the inverse of A_h and $\langle B_h=(1\cdots 1)B_h(1\cdots 1)'$. Then $28\mu(\Sigma_{2h+1}^7)=\frac{1}{2^5}(-2^2\cdots -2^2)B_h(-2^2\cdots -2^2)^t=\frac{1}{2}\langle B_h$. Obviously $B_1=A_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $\langle B_h=2$. For $h>1$, it is easy to see that

$$B_h=\left(\begin{array}{ccccc|cc} &&&&& 0 & (-1)^{h-1} \\ &&&&& 0 & 0 \\ &&&&& 0 & (-1)^{h-2} \\ &&B_{h-1}&&& 0 & 0 \\ &&&&& \vdots & \vdots \\ &&&&& 0 & (-1) \\ &&&&& 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ (-1)^{h-1} & 0 & \cdots & (-1) & 0 & 1 & 0 \end{array}\right).$$

Hence $\sphericalangle B_h = \sphericalangle B_{h-1} + 2(1+(-1)+(-1)^2+\cdots+(-1)^{h-1})$

$$= \begin{cases} \sphericalangle B_{h-1} & \text{if } h \text{ is even} \\ \sphericalangle B_{h-1}+2 & \text{if } h \text{ is odd.} \end{cases}$$

From the above we see that $\sphericalangle B_h = h$ or $h+1$ according as h is even or odd. We conclude therefore that $28\mu(\Sigma^7_{2h+1}) = [(h+1)2] \bmod 28$.

In the same way as the proof of (1), we obtain the desired result for Σ^{15}_{2h+1}.

Let $\Sigma^n \in bP_{n+1}$, and let $|\Sigma^n|$ be the order of Σ^n. Let (T, Σ^n) be a free involution on a homotopy sphere, and $\sigma(T, \Sigma^n)$ the Browder-Livesay desuspension invariant. If $\sigma(T, \Sigma^n) \neq |\Sigma^n| \bmod 2$, then (T, Σ^n) is called a *curious involution.*

Corollary 2. 2. *If* $d = \pm 3 \bmod 8$, *then* (T_d, Σ^7_d) *and* (T_d, Σ^{15}_d) *are curious involutions.*

Recently T. Yoshida [13] has shown that free involutions on 3-dimensional homology spheres satisfy that $\sigma(T, \Sigma^3) = 2\mu(\Sigma^3) \bmod 2$, where μ is the Rohlin invariant. However, the above equality need not be true for general dimensions.

3. Double suspensions of (T_d, Σ^{4k+1}_d). Let (T, Σ^n) be a free involution on a homotopy sphere. First we assume that Σ^n is diffeomorphic to the standard sphere S^n. Let $E(\Sigma^n/T)$ be the total space of disk bundle of the nontrivial line bundle over Σ^n/T. Choosing a diffeomorphism $g: \Sigma^n \longrightarrow S^n$, we attach a disk D^{n+1} to $E(\Sigma^n/T)$ via g. Then we have a homotopy projective space Q^{n+1}. Let $(\widetilde{T}, \widetilde{Q}^{n+1})$ be a free involution on the two fold cover of Q^{n+1}. We assume further that $\widetilde{Q}^{n+1}$ is diffeomorphic to S^{n+1}, then we apply the above process to construct a homotopy projective space Q^{n+2}. We call $(\widetilde{T}, \widetilde{Q}^{n+2})$ a double suspension of (T, Σ^n).

Proposition 3. 1. *The double suspension of the Brieskorn involution* (T_d, Σ^5_d) *is* (T_d, Σ^7_d), *modulo the action of* bP_8.

Proof. Let (T, Σ^7) be a double suspension of (T_d, Σ^5_d) and choose a desuspension $(T, \Sigma^6) \supset (T_d, \Sigma^5_d)$. Let (T_d, Σ^7_d) be as in Theorem 1. 1, then it has desuspensions $(T_d, \Sigma^6_d) \supset (T_d, \Sigma^5_d)$. Since $\theta^6 = 0$, Σ^6/T is diffeomorphic to Σ^6_d/T_d. The suspension construction yields that Σ^7/T is diffeomorphic to $\Sigma^7_d/T_d \# \Sigma'^7$ for some $\Sigma'^7 \in \theta^7$. So, any double suspension of (T_d, Σ^5_d) is equivariantly diffeomorphic, up to the action of bP_8, to Σ^7_d/T_d.

We note that if the differentiable structure of Σ^7 in Proposition 3. 1 is given, then we can get rid of the ambiguity of bP_8; in fact

(1) taking the two fold cover in the above proof, we can determine Σ' by $\mu(\Sigma'^7)=(\mu(\Sigma^7)-\mu(\Sigma^7_a))/2$.

(2) from the proof of Corollary 2. 11[4], bP_8 acts freely on homotopy projective 7-spaces.

By making use of the Milnor-Munkres-Novikov pairing $\rho_{n,k}: \theta^n \times \Pi_k \longrightarrow \theta^{n+k}$, where Π_k is the k-stem $\pi_{m+k}(S^m)$ for m large. Bredon [3] showed the existence of exotic actions on some elements of θ^{n+k}. We apply the result to certain homotopy projective spaces. First we note that $\theta^{14}=Z_2$ and $\Pi_1=Z_2$. We will here recall the construction for $n=14$ and $k=1$.

Let S^1 act on $S^{15}\subset R^{16}$ by means of the representation

$$\Phi: S^1=SO(2)\longrightarrow SO(16),\ \Phi(A)=\left.\begin{pmatrix} A & & \\ & \ddots & \\ & & A\end{pmatrix}\right\} 8$$

If we let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{16}$ a basis of R^{16}, then by the equivariant slice theorem we can assign to $S^1(\varepsilon_1)$ the equivariant normal framing $\mathfrak{F}$, $\langle g(\varepsilon_3), \cdots, g(\varepsilon_{16})\rangle$ $(g\in S^1)$. The principal orbit $S^1(\varepsilon_1)$ lies on the plane spanned by ε_1 and ε_2. Thus we have an element $\alpha_{15,-1}=\langle S^1(\varepsilon_1), \mathfrak{F}\rangle\in\Pi_1=\pi_{15}(S^{14})$ in Bredon's notation. He showed that $\alpha_{15,-1}\neq 0$ in Π_1. We represent $\langle S^1(\varepsilon_1), \mathfrak{F}\rangle$ by an equivariant embedding $f: S^1\times D^{14}\longrightarrow S^{15}$, where the action of S^1 is $g\times\mathrm{id}$ on $S^1\times D^{14}$ for $g\in S^1$. If we represent the generator $\sigma\in\theta^{14}$ by a diffeomorphism $h: S^{13}\longrightarrow S^{13}$, then the homotopy sphere $\rho_{14,1}(\sigma, \alpha_{15,-1})$ with an S^1-action is obtained from $S^{15}-\mathrm{int}\, f(S^1\times D^{14})$ and $S^1\times D^{14}$ by attaching via the diffeomorphism $\Psi: S^1\times S^{13}\longrightarrow f(S^1\times S^{13})$, $\Psi(x, y)=f(x, h(y))$. The S^1-action on $S^{15}-\mathrm{int}\, f(S^1\times D^{14})$ is the restriction of that on S^{15}, and on $S^1\times D^{14}$ $(x, y)\longrightarrow(gx, y)$ $(g\in S^1)$. Let P' be the Kervaire-Milnor map of θ^{15} to Π_{15}/J_{15}, where J_{15} is the image of the J-homomorphism $\pi_{15}(SO(l))\longrightarrow\pi_{15+l}(S^l)=\Pi_{15}$, l large. Then it follows from Proposition 3. 3[3] that $P'\rho_{14,1}(\sigma, \alpha_{15,-1})$ is nonzero in Π_{15}/J_{15}. Since the action of S^1 is free on S^{15}, the above action is also free on $\rho_{14,1}(\sigma, \alpha_{15,-1})$. Let a be the linear free antipodal map embedded in the S^1-action on S^{15}. We denote by T_a the corresponding Z_2-action on $\rho_{14,1}(\sigma, \alpha_{15,-1})$, and put $\rho_{14,1}(\sigma, \alpha_{15,-1})=\Sigma^{15}_a$. We note that $\Sigma^{15}_a\notin bP_{16}$ since Ker $P'=bP_{16}$. In the proof of the next proposition, the above construction will be generalized for free involutions on homotopy spheres.

Proposition 3. 2. *Let* (T, Σ^{15}) *be a free involution on a homotopy sphere, and* $g: (T, \Sigma^{15}) \longrightarrow (a, S^{15})$ *an equivariant homotopy equivalence. Then, there is a free involution* T' *on a homotopy sphere* Σ'^{15} *and an equivariant homotopy equivalence* $g': (T', \Sigma'^{15}) \longrightarrow (T_a, \Sigma_a^{15})$.

Proof. The equivariant embedding $f: S^1 \times D^{14} \longrightarrow S^{15}$ which represents the epuivariant framing $\langle S^1(\varepsilon_1), \mathfrak{F} \rangle$ induces an embedding $\bar{f}$: $P^1 \times D^{14} \longrightarrow P^{15}$. We may assume that the homotopy equivalence $\bar{g}$ of the quotient spaces: $\Sigma^{15}/T \longrightarrow P^{15}$ is transverse regular on $P^1 \overset{\bar{f}}{\subsetneq} P^{15}$. Then we have the induced embedding $\bar{f}': M^1 \times D^{14} \longrightarrow \Sigma^{15}/T$ such that $\bar{g}\bar{f}'(x, y) = \bar{f}(\bar{g}(x), y)$ for $(x, y) \in M^1 \times D^{14}$, where $M^1 = \bar{g}^{-1}(P^1)$. Define the map $t: M^1 \times D^{14} \longrightarrow S^1 \times D^{14}$ by $t(x, y) = (\bar{g}(x), y)$. By the transversality theorem, $\bar{g}$ is of degree 1, i. e., $\bar{g}_*[M^1] = [P^1]$. So t is of degree 1. As we have obtained Σ_a^{15}/T_a from P^{15}, we can construct Σ'/T' from $\Sigma^{15}/T - \text{int}\,\bar{f}'(M^1 \times D^{14})$ and $M^1 \times D^{14}$ by attaching $M^1 \times S^{13}$ to $\bar{f}'(M^1 \times S^{13})$ via the diffeomorphism $\psi_{\bar{F}'}$, $\psi_{\bar{F}'}(x, y) = \bar{f}'(x, h(y))$. The maps $\bar{g}: \Sigma^{15}/T - \text{int}\,\bar{f}'(M^1 \times D^{14}) \longrightarrow P^{15} - \text{int}\,\bar{f}(P^1 \times D^{14})$ and $t: M^1 \times D^{14} \longrightarrow P^1 \times D^{14}$ are compatible under the identifications, so that they induce a map $\bar{g}': \Sigma'^{15}/T' \longrightarrow \Sigma_a^{15}/T_a$ which is of degree 1. Since the two fold cover of Σ'^{15}/T' is $(\Sigma^{15} - \text{int}\,f'(\tilde{M}^1 \times D^{14})) \cup_{\psi'} \tilde{M}^1 \times D^{14}$ and has the homotopy type of Σ^{15}, $\bar{g}'_*$ induces an isomorphism of $\pi_1(\Sigma'^{15}/T') = Z_2$ into $\pi_1(\Sigma_a^{15}/T_a)$. Hence $\bar{g}'$ is a homotopy equivalence. Taking the two fold cover, we obtain Proposition 3. 2.

We show that (T', Σ'^{15}) constructed above is independent of the choice of equivariant homotopy equivalences of (T, Σ^{15}) to (a, S^{15}). First we prove that the above construction is compatible with normal cobordism (refer to [9] for normal cobordism).

Lemma 3. 3. *If* $F: W^{16} \longrightarrow P^{15}$ *is a normal cobordism between* f_i: $\Sigma_i/T_i \longrightarrow P^{15}$ $(i=0, 1)$, *then there is a normal cobordism* $F': W'^{16} \longrightarrow \Sigma_a^{15}/T_a$ *between* $f'_i: \Sigma_i'^{15}/T'_i \longrightarrow \Sigma_a^{15}/T_a$ $(i=0, 1)$.

Proof. Let $F: W \longrightarrow P^{15}$ be a normal map which is covered by a bundle map $B: \nu_W \longrightarrow \xi$, where ν_W is a stable normal bundle of W, and ξ a k-dimensional vector bundle, $k > 15$. We may assume that f_i is transverse regular on $P^1 \subset P^{15}$, and put $f_i^{-1}(P^1) = M_i^1$ $(i=0, 1)$. Then, by the relative transversality theorem we may assume that F is transverse regular on $P^1 \subset P^{15}$, $F^{-1}(P^1) = V^2 \subset W^{16}$ and $\partial V^2 = M_0^1 \cup M_1^1$. Since the normal bundle of V^2 is the pull-back of $\bar{f}: P^1 \times D^{14} \longrightarrow P^{15}$, we have an embedding $\bar{G}: V^2 \times D^{14} \longrightarrow W^{16}$ such that $\bar{G}|\partial V^2 \times D^{14} = \bar{f}'_0 \cup \bar{f}'_1 : M_0^1 \times D^{14} \cup M_1^1 \times D^{14} \longrightarrow \partial W^{16} = \Sigma_0^{15}/T_0 \cup \Sigma_1^{15}/T_1$, and $F\bar{G}(x, y) = \bar{f}(F(x), y)$ for $(x, y) \in V^2 \times D^{14}$. Let W'^{16} be $(W^{16} - \text{int } \bar{G}(V^2 \times D^{14})) \cup V^2 \times D^{14}$ glued by $\Phi: V^2 \times S^{13} \longrightarrow \bar{G}(V^2 \times S^{13})$, $\Phi(x, y) = \bar{G}(x, h(y))$. Then the boundary of W'^{16} consists of Σ'_i/T'_i $(i=0, 1)$ constructed in Proposition 3.2. If we define the map $s: V^2 \times D^{14} \longrightarrow P^1 \times D^{14}$ by $s(x, y) = (F(x), y)$, then as in the proof of Proposition 3.2, the maps F and s induce a degree 1 map $\bar{F}': W'^{16} \longrightarrow \Sigma_a^{15}/T_a$. Let $\xi|(P^{15} - \text{int } \bar{f}(P^1 \times D^{14}))$ be the restriction of ξ on $P^{15} - \text{int } \bar{f}(P^1 \times D^{14})$. Under the identification $\bar{\Psi}: P^1 \times S^{13} \longrightarrow \bar{f}(P^1 \times S^{13})$, $P^1 \times S^{13}$ has the bundle $\bar{\Psi}^*(\xi|\bar{f}(P^1 \times S^{13}))$. Then the obstructions to extend $\bar{\Psi}^*(\xi|\bar{f}(P^1 \times S^{13}))$ to a bundle over $P^1 \times D^{14}$ lie in the group $H^j(P^1 \times (D^{14}, S^{13}); \pi_{j-1}(SO))$. This group vanishes for $j = 14, 15$, and we can extend the bundle to a bundle ω over $P^1 \times D^{14}$. The manifold Σ_a^{15}/T_a has a bundle ξ' obtained from ξ and ω glued by $\bar{\Psi}: P^1 \times S^{13} \longrightarrow \bar{f}(P^1 \times S^{13})$. Since F is covered by the bundle map $B: \nu_W \longrightarrow \xi$, by the compatibility of s with F, there is a bundle map $B': \Phi^*(\nu_W|\bar{G}(V^2 \times S^{13})) \longrightarrow \bar{\Psi}^*(\xi|\bar{f}(P^1 \times S^{13}))$ which covers $s: V^2 \times S^{13} \longrightarrow P^1 \times S^{13}$. Let ω' be the pull back of ω by the map $s: V^2 \times D^{14} \longrightarrow P^1 \times D^{14}$, then we have a bundle map $B'': \omega' \longrightarrow \omega$ extending B'. Since $H^j(P^1 \times D^{14}, \pi_{j-1}(SO))$ is zero, ω (and hence ω') is trivial. Since any bundle over $V^2 \times D^{14}$ is trivial, we can take ω' as a normal bundle of $V^2 \times D^{14}$. Thus, B and B'' give a bundle map of $\nu_{W'}$ to ξ' which covers $\bar{F}': W' \longrightarrow \Sigma_a^{15}/T_a$. Hence $\bar{F}'$ is a normal cobordism between $\bar{f}'_i: \Sigma_i'^{15}/T'_i \longrightarrow \Sigma_a^{15}/T_a$.

Corollary 3.4, *The above construction of* (T', Σ'^{15}) *in Proposition 3.2 is independent of the choice of homotopy equivalences of* (T, Σ^{15}) *onto* (a, S^{15}).

Proof. Let $f_i: \Sigma/T \longrightarrow P^{15}$ $(i=0, 1)$ be a homotopy equivalence. We can assume that these maps are of degree 1. Then, according to Theorem II_b [10], they are homotopic. If $F: \Sigma/T \times I \longrightarrow P^{15}$ is a homotopy of these maps, then F is also of degree 1. Hence F is a homotopy equivalence, and F determines a normal map. Then it follows from Lemma 3.3 that there is a normal cobordism W between $(\bar{f}'_0, \Sigma'_0/T'_0)$ and $(f'_1, \Sigma'_1/T'_1)$. Since W has the homotopy type of $\Sigma/T \times I$, W is an h-cobordism between Σ'_0/T'_0 and Σ'_1/T'_1. Hence they are diffeomorphic.

We may write $\rho(T, \Sigma^{15}) = (T', \Sigma'^{15})$ and $\rho(g) = g'$ (see Proposition 3.2).

Proposition 3.5. *If* (T, Σ^{15}) *is a free involution on a homotopy* *sphere, then* $\rho(\rho(T, \Sigma))=(T, \Sigma))$.

Proof. Let $H: S^{13} \longrightarrow S^{13}$ be a diffeomorphism whose isotopy class is the generator of θ^{14}. Then h extends to a homotopy equivalence $\bar{h}: D^{14} \longrightarrow D^{14}$. Decompose P^{15} into $(P^{15}-\operatorname{int}\bar{f}(P^1\times D^{14}))\underset{id.}{\cup}\bar{f}(P^1\times D^{14})$, where $\bar{f}$ is the embedding introduced in the proof of Proposition 3. 2. Then there is a homotopy equivalence $\bar{f}_a: \Sigma_a^{15}/T_a \longrightarrow P^{15}$ defined by id. on $P^{15}-\operatorname{int}\bar{f}(P^1\times D^{14})$ and $\bar{f}(\mathrm{id}\times\bar{h})$ on $P^1\times D^{14}$. Let $\rho(T, \Sigma)=(T', \Sigma')$, and $\rho(\bar{g}): \Sigma'/T' \longrightarrow \Sigma_a/T_a$. Applying Proposition 3. 2 to the composition $\bar{f}_a\rho(\bar{g}): \Sigma'/T' \longrightarrow P^{15}$, we can easily see that $\rho(\overline{\bar{f}_a\rho(\bar{g})}): \Sigma/T \longrightarrow \Sigma_a^{15}/T_a$, i. e., $\rho(\rho(T, \Sigma))=(T, \Sigma)$.

Proposition 3. 6. *Let* (T, Σ^{15}) *be as in Proposition* 3. 5.

(1) ρ *commutes with the action of* θ^{15}, *i. e., if we denote by* $\bar{\rho}(\Sigma/T)$ *the quotient manifold of* $\rho(T, \Sigma)$, *then* $\bar{\rho}(\Sigma/T\#\Sigma')=\bar{\rho}(\Sigma/T)\#\Sigma'$.

(2) *let* $\rho(\Sigma)$ *be the homotopy sphere of* $\rho(T, \Sigma)$. *If* $\Sigma\in bP_{16}$ *then* $\rho(\Sigma)\notin bP_{16}$.

Proof. (1) We can do the connected sum in $\Sigma^{15}/T-\operatorname{int}\bar{f}'(M^1\times D^{14})$ under the notations used in the proof of Proposition 3.2. Hence $\bar{\rho}(\Sigma/T\#\Sigma')$ $=\bar{\rho}(\Sigma/T)\#\Sigma'$. (2) By Theorem V. 3 [9], Σ/T is normally cobordant to a homotopy equivalence $\bar{\alpha}: S^{15}/T \longrightarrow P^{15}$. Applying the Pontrjagin-Thom construction to the commutative diagram,

$$\begin{array}{ccc} S^{15} & \xrightarrow{\alpha} & S^{15} \\ \cup f' & & \cup f \\ \tilde{M}^1 & \xrightarrow{\alpha} & S^1 \end{array} \qquad \bar{\alpha}^{-1}(P^1)=M^1,$$

we obtain the commutative diagram:

$$\begin{array}{ccc} S^{15} & \xrightarrow{\alpha} & S^{15} \\ & {\scriptstyle C_T}\searrow \quad \swarrow{\scriptstyle \alpha_{15,-1}} & \\ & S^{14} & \end{array}$$

where $\alpha_{15,-1}$ is introduced in the beginning of this section. Hence α induces an isomorphism $\alpha^*: \pi_{15}(S^{14}) \longrightarrow \pi_{15}(S^{14})$ such that $\alpha^*(\alpha_{15,-1})=c_T$. The framed submanifold in S^{15}, $(\tilde{M}^1, \mathfrak{F}')=\alpha^*(S^1(\varepsilon_1), \mathfrak{F})$ determines a nonzero element in $\Pi_1=\pi_{15}(S^{14})$. Let Σ_1^{14} be the generator of θ^{14}. Then, by construction, $\rho_{14,1}(\Sigma_1^{14}, (\tilde{M}^1, \mathfrak{F}'))=\rho(T, S^{15})$. As was noted before Proposition 3. 2, $\mathrm{P}'\rho(T, S^{15})$ is nonzero in Π_{15}/J_{15}, i. e., $\rho(S^{15})\notin bP_{16}$. Since Σ/T is normally cobordant to S^{15}/T, it follows by Lemma 3. 3 that

$\bar{\rho}(\Sigma/T)$ is normally cobordant to $\bar{\rho}(S^{15}/T)$. Hence, $\rho(\Sigma)\notin bP_{16}$.

The homotopy sphere Σ_d^{15} in Theorem 1. 2 are elements of bP_{16}. Using Propositions 3. 2 and 3. 6, we shall give examples of free involutions whose homotopy spheres are not in bP_{16}.

Theorem 3. 7. *For each odd positive integer d, there exists a free involution $\rho(T_d, \Sigma_d^{15})$. If we put $\rho(T_d, \Sigma_d^{15})=(T_d', \Sigma_d'^{15})$, then $(T_d', \Sigma_d'^{15})$ has the Brieskorn involution (T_d, Σ_d^{13}) as a desuspension and $\Sigma_d'^{15}\notin bP_{16}$.*

Proof. Since (T_d, Σ_d^{15}) has the desuspensions $(T_d, \Sigma_d^{14})\supset(T_d, \Sigma_d^{13})$ in Theorem 1. 2, we have a homotopy equivalence $g: \Sigma_d^{15}/T_d \longrightarrow P^{15}$ such that $g^{-1}(P^{14})=\Sigma_d^{14}/T_d$ and $g^{-1}(P^{13})=\Sigma_d^{13}/T_d$. Let $f: P^1\times D^{14}\longrightarrow P^{15}$ be the embedding as in the proof of Proposition 3. 2. Then, from the choice of the equivariant framing $\mathfrak{F}$ defined in the beginning of this section it follows that $\bar{f}(P^1\times D^{14})$ is just the complement of the normal bundle of P^{13} in P^{15}. Thus we have the following commutative diagrams of maps;

$$\begin{array}{ccc}
\Sigma_d'^{15}/T_d' & \xrightarrow[\simeq]{\bar{\rho}(g)} & \Sigma_a^{15}/T_a \\
\cup & & \cup \\
\Sigma_d^{14}/T_d\#\Sigma_1^{14} & \xrightarrow[\simeq]{} & P^{14}\#\Sigma_1^{14} \\
\cup & & \cup \\
\Sigma_d^{13}/T_d & \xrightarrow[\simeq]{} & P^{13}
\end{array} .$$

Since $\Sigma_d^{15}\in bP_{16}$, $\rho(\Sigma_d^{15})=\Sigma_d'^{15}$ is not contained in bP_{16}.

According to the work of Kervaire-Milnor, we have a split exact sequence of θ^{15},

$$0 \longrightarrow bP_{16} \longrightarrow \theta^{15} \overset{P'}{\rightleftarrows} Z_2 \longrightarrow 0,$$

where $Z_2=\pi_{15}/J_{15}$. Denote by Σ_s^{15} the generator on the Z_2-summand, $P'(\Sigma_s^{15})\neq 0$. In the rest of this section, let (T_d, Σ_d^{15}) be a free involution on a homotopy sphere in Theorem 1. 2 and (T_d, Σ_d^{14}) its desuspension.

Lemma 3. 8. *Let Σ_1^{14} be the generator of θ^{14}. Then Σ_1^{14} acts freely on the set $\{\Sigma_d^{14}/T_d,\ d\ odd\}$, i. e., $\Sigma_{d_0}^{14}/T_{d_0}\#\Sigma_1\neq\Sigma_{d_1}^{14}/T_{d_1}$ for arbitrary d_0, d_1.*

Proof. If $\Sigma_{d_0}^{14}/T_{d_0}$ is diffeomorphic to $\Sigma_{d_1}^{14}/T_{d_1}\#\Sigma_1^{14}$, then the suspension construction yields that $\Sigma_{d_0}^{15}/T_{d_0}$ is diffeomorphic to $\Sigma'^{15}_{d_1}/T'_{d_1}\#\Sigma'$ for some $\Sigma'\in\theta^{15}$. Taking the two fold cover, we see that $\Sigma_{d_0}^{15}$ is diffeomorphic to $\Sigma'^{15}_{d_1}\#2\Sigma'$. Thus by Theorem 3. 7 we have $0=P'(\Sigma_{d_0}^{15})=P'(\Sigma'^{15}_{d_1})+2P'(\Sigma')=P'(\Sigma'^{15}_{d_1})\neq 0$, which is a contradiction.

Lemma 3. 9. *Σ_s^{15} acts freely on the set $\{\Sigma_d^{15}/T_d,\ d$ odd$\}$.*

In advance of proving the lemma, we seek the specific geometry of the desuspensions of (T_d, Σ_d^{15}) $(=\partial(T_d, M_d^{16}))$. Recall that $(T_d, \Sigma_d^{13})=\partial(T_d, M_d^{14})$ in the proof of Theorem 1. 2. We note that M_d^{14} is 6-connected. The generators of $H_7(M_d^{14})$ are taken to be a symplectic basis $\{\alpha_i, \beta_i\}$ $(i=1, \cdots, (d-1)/2)$. Since $\pi_6(SO(7))=0$, we can perform a surgery on the elements α_i's. Let C^{15} be a trace between M_d^{14} and a homotopy disk rel. boundary. We put $(T_d, V_d^{15})=(C^{15}\cup C^{*15})$, glued on (T_d, M_d^{14}) equivariantly, where C^* is a copy of C. Next we perform a surgery on the elements β_i's to yield a trace F^{15} between M_d^{14} and a homotopy disk E^{14} rel. boundary. Then it follows from the Mayer-Vietoris exact sequence that the manifold $(C^{15}\cup F^{15})$ glued along M_d^{14} is the 15-disk. M_d^{14} has the trivial normal bundle in V_d^{15}, i. e., $M_d^{14}\times I\subset V_d^{15}$. If we add $F^{15}\times I$ to $V_d^{15}\times I$ along $(M_d^{14}\times I)\times 1$, then we have a trace X_d^{16} between V_d^{15} and a homotopy disk E^{15}, where $E^{15}=(C\cup F)\cup(C^*\cup F^*)$ glued along E^{14} (see Figure 1). It is easy to see that X_d^{16} is 7-connected. Arranging X_d^{16}, we have a 7-connected 16-manifold Y_d^{16} whose boundary is a union of a homotopy disk D_d^{15} and V_d^{15}, glued along ∂V_d^{15} (see Figure 2).

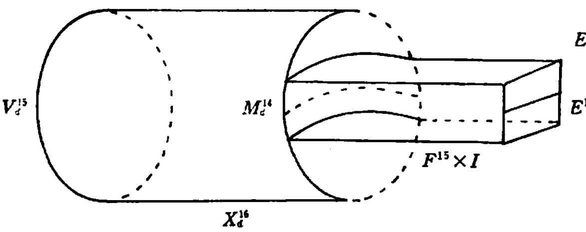

Figure 1

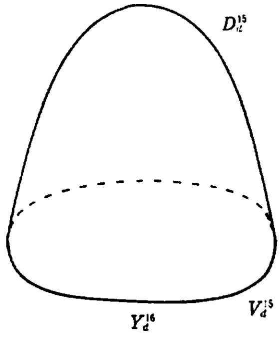

Figure 2

We put $(T_d, W_d^{16})=Y_d^{16}\cup Y_d^{*16}$, glued on (T_d, V_d^{15}) equivariantly, where Y_d^* is a copy of Y_d. Then the manifold W_d^{16} is 7-connected, so $\partial W_d^{16}\in bP_{16}$. Since $(T_d, \partial W_d^{16})$ has the desuspension $(T_d, \partial V_d^{15})$, we see by Lemma 3. 8 that $\partial V_d^{15}/T_d\cong\Sigma_d^{14}/T_d$.

Decompose D_d^{15} into $D_0^{15} \cup K_d^{15}$, where D_0^{15} is the disk with boundary S_0^{14} and K_d^{15} is an h-cobordism between S_0 and Σ_d^{14}. Let Z_d^{16} be any 7-connected 16-manifold with boundary $(V_d^{15} \cup E_d^{15})$ glued along Σ_d^{14}, where E_d^{15} is any homotopy disk with boundary Σ_d^{14}. Then by the h-cobordism theorem, there is a diffeomorphism $\psi_d : K_d^{15} \cup E_d^{15}$ (glued along Σ_d^{14}) $\longrightarrow D_0^{15}$. Since $D_d^{15} \cup E_d^{15}$ (glued along Σ_d^{14}) bounds the 7-connected 16-manifold $(Y_d^{16} \cup Z_d^{16})$ glued along V_d^{15}, we see by Lemma 7 [11] that $\psi_d | \partial(K_d^{15} \cup E_d^{15}) = \psi_d | S_0 : S_0^{14} \longrightarrow S_0^{14}$ is an element in bP_{16}.

Proof of Lemma 3.9. Suppose that there is a diffeomorphism f of $\Sigma_{d_2}^{15}/T_{d_2}$ onto $\Sigma_{d_1}^{15}/T_{d_1} \# \Sigma_s^{15}$. In the two fold cover, there is the diffeomorphism $\bar{f} : D_{d_2}^{15} \longrightarrow D_{d_1}'^{15}$ induced from the covering diffeomorphism $\bar{f}$ of f, where $\partial D_{d_1}'^{15} = \Sigma_{d_1}^{14}$. Decompose $D_{d_1}'^{15}$ into $D_0^{15} \cup K_{d_1}'^{15}$ as above. Then we may assume that $\bar{f} : K_{d_2} \longrightarrow K'_{d_1}$ is a diffeomorphism such that $\bar{f} | S_0 = \mathrm{id}$. If we attach $E_{d_2}^{15}$ to K'_{d_1} via $\bar{f} : \Sigma_{d_2}^{14} \longrightarrow \Sigma_{d_1}^{14}$, then the manifold $(D_0^{15} \cup K_{d_1}'^{15}) \cup E_{d_2}^{15}$ bounds the 7-connected 16-manifold $(Y_{d_2}^{16} \cup Z_{d_2}^{16})$ followed by $\bar{f}$. Again, by the h-cobordism theorem, there is a diffeomorphism $\Phi : K'_{d_1} \cup E_{d_2}^{15}$ (glued by $\bar{f}$) $\longrightarrow D_0^{15}$ such that $\Phi | \partial(K'_{d_1} \cup E_{d_2}^{15}) = \Phi | S_0 \in bP_{16}$. On the other hand, K'_{d_1} is obtained from $S_0^{14} \times I \cup K_{d_1}$, glued on the diffeomorphism $\alpha : S_0^{14} \times 1 \longrightarrow S_0^{14}$ which represents Σ_s^{15}. Thus $\Phi | S_0$ is the composition of α and the diffeomorphism of $\partial(K_{d_1} \cup_{\bar{f}} E_{d_2}^{15})$ $(=S_0)$ onto S_0, which is an element of bP_{16} by the argument before the proof of Lemma 3.9. Hence, α which represents Σ_s^{15} is contained in bP_{16}. This yields a contradiction.

Corollary 3.10. *Let* $(T'_d, \Sigma_d'^{15})$ *be a free involution on a homotopy sphere in Theorem* 3.7. *Then,* Σ_s^{15} *acts freely on the set* $\{\Sigma_d'^{15}/T'_d,\ d \text{ odd}\}$.

Proof. Suppose that $\Sigma_{d_1}'^{15}/T'_{d_1} \# \Sigma_s^{15} \cong \Sigma_{d_2}'^{15}/T'_{d_2}$. Then, by Theorem 3.7, $\bar{\rho}(\Sigma_{d_1}^{15}/T_{d_1}) \# \Sigma_s \cong \bar{\rho}(\Sigma_{d_2}^{15}/T_{d_2})$. It follows by Proposition 3.6 that $\bar{\rho}(\Sigma_{d_1}^{15}/T_{d_1}) \# \Sigma_s = \bar{\rho}(\Sigma_{d_1}^{15}/T_{d_1} \# \Sigma_s)$, and hence $\Sigma_{d_1}^{15}/T_{d_1} \# \Sigma_s \cong \Sigma_{d_2}^{15}/T_{d_2}$. This contradicts Lemma 3.9.

Theorem 3.11. *The set of double suspensions of the Brieskorn involutions* (T_d, Σ_d^{13}) *consists of exactly four distinct elements, modulo the action of* bP_{16}, *i.e., the quotient manifolds* Σ_d^{15}/T_d, $\Sigma_d^{15}/T_d \# \Sigma_s^{15}$, $\Sigma_d'^{15}/T'_d$ *and* $\Sigma_d'^{15}/T'_d \# \Sigma_s^{15}$.

Proof. Let (T, Σ^{15}) be a double suspension of (T_d, Σ_d^{13}) and

$(T, \Sigma^{14}) \supset (T_d, \Sigma_d^{13})$ a desuspension of (T, Σ^{15}). Then the suspension construction yields that (i) $\Sigma^{14}/T \cong \Sigma_d^{14}/T_d$ or (ii) $\Sigma^{14}/T \cong \Sigma_d^{14}/T_d \# \Sigma_1^{14}$. There follows that Σ^{15}/T is diffeomorphic to $\Sigma_d^{15}/T_d \# \Sigma'$ or $\Sigma_d'^{15}/T_d' \# \Sigma'$ for some $\Sigma' \in \theta^{15}$ according as (i) or (ii). Then by Lemma 3. 9 and Corollary 3. 10, Σ^{15}/T is diffeomorphic to one of the elements in the theorem, modulo the action of bP_{16}.

4. Application. First we determine the periodicity of the Brieskorn involutions.

Proposition 4. 1. (1) *If* $\Sigma_d^5/T_d \cong \Sigma_{d'}^5/T_{d'}$, *then* $d \equiv \pm d' \bmod 2^4$.
(2) *If* $\Sigma_d^{13}/T_d \cong \Sigma_{d'}^{13}/T_{d'}$, *then* $d \equiv \pm d' \bmod 2^8$.

Proof. From the discussion of the preceding section, the suspension construction yields that $\Sigma_d^{4k+3}/T_d \cong \Sigma_{d'}^{4k-3}/T_{d'} \# \Sigma'$ for some $\Sigma' \in bP_{4k+4}$. The equivariant connected sum does not affect the spin invariant provided Σ' bounds a spin manifold. Hence we conclude that $a(T_d, \Sigma_d^{4k+3}) = a(T_{d'}, \Sigma_{d'}^{4k+3})$. So by Theorem 1. 2 we have $d \equiv \pm d' \bmod 2^{2k+2}$. This result improves the earlier estimate for the spin invariants for the Brieskorn involutions (see [8] p. 337).

The following corollary gives an answer to the question raised in [8, p. 338] which is concerning with the classification of $hS(P^5)$.

Corollary 4. 2. *Put* $\Pi_d^{4k+1} = \Sigma_d^{4k+1}/T_d$.
(1) $hS(P^5) = \{\Pi_d^5, d = 1, 3, 5, 7\}$, *and* $\Pi_d^5 \cong \Pi_{-d+2^4}^5 \cong \Pi_{d+2^4}^5$ *for each odd* $d > 0$.
(2) $\Pi_d^{13} \cong \Sigma_{-d+2^8}^{13} \cong \Sigma_{d+2^8}^{13}$ *for each odd* $d > 0$.

Proof. We note first that Σ_d^{4k+1} admits an orientation reversing diffeomorphism. Since $hS(P^5) = [P^5, G/O] = Z_4$ ([9]), (1) is obtained by Proposition 4. 1. For (2), note that the number of distinct elements of the set $\{\Sigma_d^{13}/T_d, d > 0\}$ is at most 2^6 by the results on $KO(P^{13})$ ([7]).

Let N_h^7 be the Milnor sphere which is the boundary of the D^4-bundle $\xi_{h,1-h}$ over S^4 [6]. Taking the antipodal map on each fiber, we obtain a smooth free involution $\alpha_h : N_h^7 \longrightarrow N_h^7$. In [6], it has been shown that (α_h, N_h^7) has a double suspension. Let (α_h, N_h^5) be its 5-dimensional desuspension. We correct here the assertion of these desuspensions [12].

Proposition 4. 3. (1) (α_h, N_h^7) *has* $(T_{2h-1}, \Sigma_{2h-1}^5)$ *as a 5-dimensional desuspension, i. e.,* $(\alpha_h, N_h^5) \cong (T_{2h-1}, \Sigma_{2h-1}^5)$.

(2) $(T_{2h+1}, \Sigma^7_{2h+1}) \cong (\alpha_{h+1}, N^7_{h+1}) \underset{Z_2}{\#} \Sigma'_h$, *where* Σ'_h *is determined by* $\mu(\Sigma'_h) = ([(h+1)/2] - h(h-1)/2)/2$. *Hence* Π^5_d *is DIFF-exotic if* $d=3, 5$ mod 8.

Proof. Let $\xi_{h,1-h}$ be the bundle induced from the map $f_{h,1-h}$ and α'_h the antipodal map on the fiber, $\alpha'_h | N^7_h = \alpha_h$, then (α_h, N^7_h) bounds the spin manifold $(\alpha'_h, E(\xi_{h,1-h}))$. Obviously, Fix $(\alpha'_h, E(\xi_{h,1-h}))$ is the zero section S^4. Since the Pontrjagin class $P_1(\xi_{h,1-h})$ is $\pm 2(2h-1)\iota$, by definition [2] we have $a(\alpha_h, N^7_h) = \pm(2h-1)$ mod 2^4. Suppose $(\alpha_h, N^5_h) \cong (T_d, \Sigma^5_d)$ for some d. As in the proof of Proposition 4. 1, we can see $2h-1 = \pm d$ mod 2^4. We are free to take d within $\{\pm d \bmod 2^4\}$. So, in particular our assertion follows by taking $d=2h-1$. Since $(T_{2h+1}, \Sigma^7_{2h+1})$ and $(\alpha_{h+1}, N^7_{h+1})$ have the same characteristic submanifold $(T_{2h+1}, \Sigma^5_{2h+1})$, we obtain the required diffeomorphism by the construction of suspensions. The rest of proposition follows from the fact that $28\mu(N^7_h) = h(h-1)/2$. If $h=2, 3$ mod 4, then $28\mu(N^7_h)$ is odd, and so its desuspension Π^5_d $(d=2h-1)$ is DIFF-exotic if $d=3, 5$ mod 8.

Under the above situation, if we pattern after the argument of the assertion in [12], we can construct an equivariant diffeomorphism λ: $(\alpha_h, N^5_h) \longrightarrow (B_{h-1}, \partial M^6_{h-1})$ $(=(T_{2h-1}, \Sigma^5_{2h-1}))$ actually for each h.

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