Math. J. Okayama Univ. 23 (1981), 91—98

ON CONFORMAL DIFFEOMORPHISMS WITH DECOMPOSABLE ASSOCIATED SCALAR FIELD

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Introduction. In a recent paper [4], one of the present authors has shown that there is no global conformal diffeomorphism between complete product Riemannian manifolds M and M^* such that the product structures F and G of them are not commutative under it in a dense subset of M, and given an example of a global conformal diffeomorphism commuting the product structures. There remain some open problems concerning this, for instance, whether the condition "in a dense subset" of the theorem can be replaced with "in an open subset" or not.

Our purpose of the present paper is to show that the replacement is possible in the case where the scalar field ρ associated with the conformal diffeomorphism is *decomposable* in an open subset of $M = M_1 \times M_2$ and depends on points of obth M_1 and M_2, that is, ρ is the sum

$$\rho = \rho_1 + \rho_2$$

of non-constant functions ρ_1 on M_1 and ρ_2 on M_2.

After preliminaries are given in § 1, we shall prove in § 2 that the parts ρ_1 and ρ_2 are special concircular scalar fields in M_1 and M_2 respectively. In § 3 we shall give expressions of ρ with respect to adapted coordinate systems in $M = M_1 \times M_2$. Then, in § 4, we shall show that no global conformal diffeomorphism with such a solution of ρ can be admitted between complete product Riemannian manifolds M and M^*.

1. Preliminaries. Let $M = M_1 \times M_2$ and $M^* = M_1^* \times M_2^*$ be product Riemannian manifolds of dimension $n \geqq 3$, and denote the structures by (M, g, F) and (M^*, g^*, G) where g and g^* are the metric tensors and F and G the product structures of M and M^* respectively. Throughout this paper, we shall assume that the manifolds are connected and the differentiability is of class C^∞. The dimensions n_1 and n_2 of the parts M_1 and M_2 may be different from those of M_1^* and M_2^*. Greek indices run on the range 1 to n, and Latin indices on the following ranges:

$$h, \ i, \ j = 1, \ 2, \ \cdots, \ n_1,$$
$$p, \ q, \ r = n_1 + 1, \ \cdots, \ n,$$

respectively. Summation convention is applied to repeated indices on their own ranges.

We shall use a separate coordinate system $(x^\iota) = (x^h, x^p)$ in $M = M_1 \times M_2$ with respect to the product structure F. The metric tensor g has components

$$(g_{\mu\lambda}) = \begin{pmatrix} g_{ji} & 0 \\ 0 & g_{rq} \end{pmatrix},$$

where g_{ji} depend on the coordinates (x^h) of M_1 only and g_{rq} on (x^p) of M_2 only. The Christoffel symbol and the curvature tensor of M have pure components only, and the covariant differentiations ∇_i along M_1 and ∇_q along M_2 commute with one another.

A conformal diffeomorphism f of M to M^* is characterized by the metric change

$$f^*(g^*) = \frac{1}{\rho^2} g,$$

and ρ is called the *associated* scalar field with f. Hereafter, the image of a quantity of M^* to M by the induced map f^* of f will be denoted by the same letter as the original one, for example, we write g^* for $f^*(g^*)$ and G for $f^*(G)$. If $FG = GF$ at a point $P \in M$, then we say that the structures F and G are *commutative* at P under f. The commutativity is equivalent to the purity of G with respect to F. We put

$$G_{\mu\lambda} = G_\mu^\kappa g_{\kappa\lambda},$$

which is symmetric in λ and μ.

Let Y be the gradient vector field (ρ^κ) of ρ, and Y_1 and Y_2 the components (ρ^h) and (ρ^p) of Y belonging to M_1 and M_2 respectively. We denote by Φ the squared length of Y:

$$\Phi = |Y|^2 = \rho_\kappa \rho^\kappa,$$

and put the open subset U as

$$U = \{P \mid Y_1(P) \neq 0 \text{ and } Y_2(P) \neq 0\}.$$

It is proved in [4] that the product structures F and G are not commutative under f in U, and we have the equations

$$(1.1) \qquad \begin{cases} \nabla_j \rho_i = \phi_1 g_{ji} + \dfrac{C}{\rho} G_{ji}, \\ \nabla_q \rho_i = \qquad\quad \dfrac{C}{\rho} G_{qi}, \\ \nabla_q \rho_p = \phi_2 g_{qp} + \dfrac{C}{\rho} G_{qp} \end{cases}$$

in every connected component of U, where C is a constant and the coefficients ϕ_1 and ϕ_2 are functions satisfying the relation

$$\phi_1 + \phi_2 = \frac{\phi}{\rho} = \frac{1}{\rho} \rho_\varepsilon \rho^\varepsilon . \tag{1.2}$$

Moreover we have seen that we may put

$$\phi_1 - \phi_2 = k\rho , \tag{1.3}$$

k being a constant. The constants C and k might be different from one connected component of U to another.

In general, a scalar field ρ in a Riemannian manifold M is said to be *concircular* if it satisfies the equation

$$\Gamma_\mu \rho_\lambda = \phi g_{\mu\lambda} , \tag{1.4}$$

and to be *special concircular* if it satisfies the equation

$$\Gamma_\mu \rho_\lambda = (k\rho + b) g_{\mu\lambda} , \tag{1.5}$$

k and b being constants. Properties of concircular scalar fields play important roles, and we refer to [2] and [3] as to them. The trajectories of the gradient vector field $Y = (\rho^\varepsilon)$ are geodesics, called *ρ-curves*, and, in a neighborhood of an ordinary point of ρ, there is a local coordinate system, called an *adapted* one, such that the first coordinate x is the arc-length of ρ-curves, ρ is a function of x only, and the metric form ds^2 of M is given in the form

$$ds^2 = ds^2 + (\rho'(x))^2 \overline{ds^2},$$

where prime indicates derivative in x and $\overline{ds^2}$ is the metric form of an $(n-1)$-dimensional Riemannian manifold $\overline{M}$, see also [1]. Along the ρ-curves, the equations (1. 4) and (1. 5) reduce to the ordinary differential equations

$$\rho''(x) = \phi$$

and

$$\rho''(x) = k\rho + b .$$

2. The decomposable associated scalar field. Now we suppose that, in an open subset U' of U, the associated scalar field ρ is the sum

$$\rho = \rho_1 + \rho_2 \tag{2.1}$$

of non-constant functions ρ_1 depending on (x^h) only and ρ_2 depending on (x^p) only. Then we have

$$\rho_i = \Gamma_i \rho_1, \quad \rho_q = \Gamma_q \rho_2$$

and

$$\Gamma_q \Gamma_i \rho = 0$$

in the open subset U'. Since there is a hybrid component $G_{qi} \neq 0$, it follows from the equation (1. 1, 2) that $C = 0$ and ρ is decomposable in the connected component of U containing U'. Consequently we may suppose that the open subset U' is the connected component of U.

Then the equations (1. 1, 1) and (1. 1, 3) turn to

$$\begin{cases} \Gamma_j \rho_i = \phi_1 g_{ji}, \\ \Gamma_q \rho_p = \phi_2 g_{qp} \end{cases} \tag{2. 2}$$

in U'. Therefore the function ϕ_1 depends on (x^h) only and ϕ_2 on (x^p) only. Substituting (2. 1) into (1. 3), we have the equation

$$\phi_1 - k\rho_1 = \phi_2 + k\rho_2.$$

Since the left hand side depends on (x^h) only and the right hand side on (x^p) only, both sides are equal to a constant, say b. Hence the functions ϕ_1 and ϕ_2 are given by

$$\phi_1 = k\rho_1 + b, \quad \phi_2 = -k\rho_2 + b, \tag{2. 3}$$

and the equations (2. 2) become

$$\begin{cases} \Gamma_j \Gamma_i \rho_1 = (k\rho_1 + b)\, g_{ji}, \\ \Gamma_q \Gamma_p \rho_2 = (-k\rho_2 + b)\, g_{qp}. \end{cases} \tag{2. 4}$$

If we denote by $M_1(P)$ and $M_2(P)$ the parts of M passing through a point $P \in M$, then these equations mean that the parts ρ_1 and ρ_2 of ρ are non-constant special concircular scalar fields in the intersections $U' \cap M_1(P)$ and $U' \cap M_2(P)$ respectively. Since there are at most two isolated stationary points of a concircular scalar field, the closure of the intersection $U' \cap M_1(P)$ in $M_1(P)$ coincides with $M_1(P)$ provided $n_1 \geqq 2$ and the closure of $U' \cap M_2(P)$ in $M_2(P)$ with $M_2(P)$ provided $n_2 \geqq 2$. Therefore the open subset U consists of one component U', and the closure of U coincides with the manifold M. If one of the parts, say M_1, is of dimension 1, then the part ρ_1 of ρ is given by a hyperbolic or sine function in $U' \cap M_1(P)$, as will be seen later in the equations (3. 6) and (3. 9). Hence the stationary points of ρ_1 are isolated in M_1 by means of differentiability of ρ_1, and the closure of U coincides with the manifold M. Therefore, in any case, the equations (2. 4) are valid over the whole manifold $M = M_1 \times M_2$. Thus we can state the following

Lemma 1. *Let M and M^* be product Riemannian manifolds and ρ the scalar field associated with a non-homothetic conformal diffeomorphism f of M to M^*. If ρ is decomposable in an open subset in $M = M_1 \times M_2$, and the parts ρ_1 and ρ_2 of ρ are not constants in the subset, then ρ is globally decomposable and the parts ρ_1 and ρ_2 are special concircular scalar fields in M_1 and M_2 satisfying the equations* (2. 4) *respectively.*

3. Expressions of the associated scalar field ρ. We shall seek for expressions of ρ in all possible cases under the assumptions of Lemma 1.

In the case $k = 0$, the equations (2. 4) become together the tensor equation

$$\Gamma_\mu \rho_\lambda = b g_{\mu\lambda} \tag{3. 1}$$

and the gradient vector field Y is parallel if $b = 0$ and concurrent if $b \neq 0$. Along any geodesic curve with arc-length x, the equation (3. 1) reduces to the ordinary differential equation

$$\rho'' = b. \tag{3. 2}$$

By choosing suitably the arc-length x of the ρ-curves, ρ is given by

$$\rho = \begin{cases} ax & (b = 0), \\ \frac{1}{2} bx^2 + a & (b \neq 0), \end{cases} \tag{3. 3}$$

a being a constant, and the metric form ds^2 of M is expressed as

$$ds^2 = \begin{cases} dx^2 + \overline{ds}^2 & (b = 0), \\ dx^2 + x^2\overline{ds}^2 & (b \neq 0), \end{cases} \tag{3. 4}$$

in the respective cases.

In the case $k \neq 0$, we may put $k = c^2$, c being a positive constant, without loss of generality. The equation (2. 4, 1) reduces to the ordinary differential equation

$$\rho_1''(x) = c^2 \rho_1 + b \tag{3. 5}$$

along any geodesic with arc-length x in M_1. By choosing suitably the arc-length x of the ρ-curves of ρ_1, the part ρ_1 is given by

$$\rho_1 = \begin{cases} (\mathrm{a}) & a_1 \exp\, cx - b/c^2\,, \\ (\mathrm{b}) & a_1 \sinh\, cx - b/c^2\,, \\ (\mathrm{c}) & a_1 \cosh\, cx - b/c^2\,, \end{cases} \tag{3.6}$$

a_1 being a non-zero constant. In an adapted coordinate system in M_1, the metric form ds_1^2 of M_1 is expressed as

$$ds_1^2 = \begin{cases} (\mathrm{a}) & dx^2 + (\exp 2cx)\overline{ds_1^2}\,, \\ (\mathrm{b}) & dx^2 + (\cosh cx)^2\overline{ds_1^2}\,, \\ (c) & dx^2 + (\sinh cx)^2\overline{ds_1^2} \end{cases} \tag{3.7}$$

in the respective cases of (3.6), where $\overline{ds_1^2}$ is the metric form of an $(n_1 - 1)$-dimensional Riemannian manifold $\overline{M}_1$.

On the other hand, the equation (2.4, 2) reduces to the ordinary differential equation

$$\rho_2''(y) = -c^2\rho_2 + b \tag{3.8}$$

along any geodesic with arc-length y in M_2. By choosing suitably the arc-length y of the ρ-curves of ρ_2, the part ρ_2 is given by

$$\rho_2 = a_2 \cos\, cy + b/c^2\,, \tag{3.9}$$

a_2 being a non-zero constant. In an adapted coordinate system in M_2, the metric form ds_2^2 of M_2 is expressed as

$$ds_2^2 = dy^2 + (\sin cy)^2\overline{ds_2^2}\,, \tag{3.10}$$

where $\overline{ds_2^2}$ is the metric form of an $(n_2 - 1)$-dimensional manifold $\overline{M}_2$.

By adding the expressions (3.6) and (3.9), we see that associated scalar field ρ is given by

$$\rho = \begin{cases} (\mathrm{a}) & a_1 \exp\, cx + a_2 \cos\, cy\,, \\ (\mathrm{b}) & a_1 \sinh\, cx + a_2 \cos\, cy\,, \\ (\mathrm{c}) & a_1 \cosh\, cx + a_2 \cos\, cy \end{cases} \tag{3.11}$$

in the respective cases.

4. Theorem. We recall the following lemma [4, Lemma 5] for later use:

Lemma 2. *Let M and M^* be complete Riemannian manifolds and f a diffeomorphism of M onto M^*. If the length of a differentiable curve Γ in M is bounded, then so is the length of the image $\Gamma^* = f(\Gamma)$ in M^*.*

Now we suppose that the manifold M and M^* are complete and f a global conformal diffeomoprhism of M onto M^*.

In the case $k = 0$ and $b = 0$, M is globally the product of a straight line I with an $(n-1)$-dimensional Riemannian manifold $\overline{M}$. Since the arc-length x of I is extendable to the infinity in a complete manifold, the associated scalar field ρ given by the expression (3. 3, 1) vanishes at the point of I corresponding to $x = 0$. This contradicts to the positiveness of ρ.

In the case $k = 0$ and $b \neq 0$, the constants a and b in the expression (3. 3, 2) should be positive because ρ is positive for all value of x. The point O corresponding to $x = 0$ is the stationary one of ρ, the $(n-1)$-dimensional Riemannian manifold $\overline{M}$ with metric form $\overline{ds}^2$ is of constant curvature 1, and M itself a Euclidean space. The rays issuing from O are ρ-curves. The arc-length s^* of the image Γ^* of a ρ-curve Γ in M under conformal diffeomorphism f is related to the arc-length x by

$$\frac{ds^*}{dx} = \frac{1}{\rho} = \frac{2}{bx^2 + a}.$$

Putting $s^* = 0$ corresponding to $x = 0$, we have

$$s^* = \frac{2}{\sqrt{ab}} \arctan \sqrt{\frac{b}{a}}\, x .$$

and see

$$s^* \longrightarrow \frac{\pi}{\sqrt{ab}} \qquad (x \to \infty).$$

This means that the length of the image Γ^* is bounded and it is a contradiction by means of Lemma 2.

In the case $k = c^2 \neq 0$, the arc-lengths x and y are extendable to the infinity. In the first case (a) and the second (b) of (3. 11), ρ has zero points. In the third case (c) with $|a_1| \leqq |a_2|$, ρ has zero points too. Thus in these cases there is no global conformal diffeomorphism of M to M^*.

In the third case (c) with $|a_1| > |a_2|$, the constant a_1 should be positive. Let P be a point corresponding to $y = \pi/2c$, and Γ a ρ-curve lying in the part $M_1(P)$ passing through P. The arc-length s^* of the image Γ^* is related to x by the equation

$$\frac{ds^*}{dx} = \frac{1}{a_1 \cosh cx}.$$

Integrating this equation, we have

$$s^* - s_0^* = \frac{2}{a_1 c} \arctan(\exp cx) - \frac{\pi}{2a_1 c},$$

where s_0^* is the value of s^* corresponding to the point P of ρ_1 on Γ, and see

$$s^* - s_0^* \longrightarrow \frac{\pi}{2a_1 c} \qquad (x \to \infty).$$

This implies the boundedness of the length of the image Γ^* and leads to a contradiction by means of Lemma 2. Thus we have established the following

Theorem. *Let M and M^* be complete product Riemannian manifolds. Then there is no global conformal diffeomorphism of M onto M^* such that the associated scalar field ρ is decomposable, $\rho = \rho_1 + \rho_2$, in an open subset U of M and the parts ρ_1 and ρ_2 are not constants in U.*

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(*Received June 17, 1980*)