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ON THE EULER-POINCARÉ CHARACTERISTIC OF 6-DIMENSIONAL HARMONIC MANIFOLDS

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1. Introduction. An analytic Riemannian manifold M is said to be harmonic at a point m of M, if Δs is a function of s only, where Δ denotes the Laplacian of M and s the geodesic distance from m to the point in a normal neighborhood of m. If M is harmonic at any point, M is said to be harmonic. For a harmonic manifold, it is known that $\Delta\Omega = f(\Omega)$ is a function of Ω only and does not depend on the reference point m, where $\Omega :=s^2/2$. Then $f(\Omega)$ is called the characteristic function of M.

The purpose of this note is to estimate the Euler-Poincaré characteristic $\chi(M)$ of a 6-dimensional compact harmonic Riemannian manifold M in terms of some curvature tensors and the volume $\mathrm{vol}\, M$ of M. We denote by $R=(R^i_{jkl})$[1], $\rho = (R^r_{jkr}) = (R_{jk})$ and τ the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively. The results are stated as follows:

Theorem 1. *Let M be a 6-dimensional compact orientable harmonic Riemannian manifold, and f the characteristic function of M. Then the following inequality holds*

$$\chi(M) \leqq -\frac{54\,\mathrm{vol}\, M}{384\pi^3}(5\dot f(0)^3 + 82\dot f(0)\ddot f(0) + 140\dddot f(0)),$$

and the equality holds if and only if M is locally symmetric.

Theorem 2. *Let M be a 6-dimensional compact harmonic Kaehler manifold.*

(1) *If $\tau > 0$, then M is of constant holomorphic sectional curvature.*

(2) *If $\tau = 0$, then $\chi(M) = 0$.*

(3) *If $\tau < 0$, then the following inequality holds*

$$\chi(M) \geqq \frac{\tau^3\,\mathrm{vol}\, M}{12\cdot 384\pi^3},$$

1) We follow the definition of the Riemannian curvature tensor in [11], which is different from that of [2] and [9] in sign.

and the equality holds if and only if M is of constant holomorphic sectional curvature.

Our theorems are related to the conjecture that any harmonic Riemannian manifold is locally symmetric (see [2], [8]).

Example. Let S^6 be a 6-dimensional sphere with constant sectional curvature k, and PC^3 a 6-dimensional complex projective space with constant holomorphic sectional curvature k. Then the characteristic function of S^6 and PC^3 are respectively given by $f(\Omega)=1+5\sqrt{2k\Omega}\cot\sqrt{2k\Omega}$ and $f(\Omega)=1+5l\cot l-l\tan l$, where $l:=\sqrt{k}s/2$ ([8], [10], [12] and also §4). By a straightforward calculation, we obtain the extremal case in Theorem 1.

2. Proof of Theorem 1. By the harmonicity of M, M is Einstein and the following equalities hold ([2], [8], [11]):

$$\tau=-9\dot{f}(0),\quad |R|^2=-9\{\dot{f}(0)^2+20\ddot{f}(0)\}, \tag{2.1}$$

$$27|\nabla R|^2-\frac{8}{9}\tau^3-24\tau|R|^2+112\beta-32\gamma=151200\dddot{f}(0), \tag{2.2}$$

where $\nabla R:=(R^i_{jkl,h})$, $\beta:=R^{ijkl}R_{ij}{}^{uv}R_{kluv}$, $\gamma:=R^{ijkl}R_i{}^u{}_k{}^v R_{julv}$ and $|\cdot|$ denotes the norm. By making use of $\Delta|R|^2=0$ and the well-known Lichnerowicz formula, we have

$$|\nabla R|^2+\frac{1}{3}\tau|R|^2+\beta+4\gamma=0. \tag{2.3}$$

In the proofs of our theorems, we shall use the following expression for Gauss-Bonnet theorem ([9]).

Lemma 1. *Let M be a 6-dimensional compact orientable Riemannian manifold. Then*

$$\begin{aligned}\chi(M)=\frac{1}{384\pi^3}\int_M\{&\tau^3-12\tau|\rho|^2+3\tau|R|^2+16R_i{}^jR_j{}^kR_k{}^i\\ &-24R^{ik}R^{jl}R_{ijkl}-24R^{rs}R_{rjkl}R_s{}^{jkl}+8\gamma-4\beta\}\,dM,\end{aligned} \tag{2.4}$$

where dM denotes the volume element of M.

Since M is Einstein, (2.4) takes the following form

$$\chi(M)=\frac{1}{384\pi^3}\int_M\left\{\frac{\tau^3}{9}-\tau|R|^2+8\gamma-4\beta\right\}dM. \tag{2.5}$$

Eliminating β, γ from (2.2), (2.3) and (2.5), we have

$$(2.6)\quad \begin{aligned} &20\cdot 384\pi^3\chi(M)-\int_M\left\{\frac{4}{3}\tau^3-\frac{164}{3}\tau|R|^2-151200\ddot{f}(0)\right\}dM \\ &= -5\int_M |\nabla R|^2\, dM. \end{aligned}$$

From (2. 1) and (2. 6) we have

$$\begin{aligned} &\chi(M)+\frac{54}{384\pi^3}\int_M \{5\dot{f}(0)^3+82\dot{f}(0)\ddot{f}(0)+140\dddot{f}(0)\}\,dM \\ &= -\frac{1}{4\cdot 384\pi^3}\int_M |\nabla R|^2\, dM, \end{aligned}$$

whence it follows Theorem 1.

3. Proof of Theorem 2. Let M be a $2n$-dimensional compact Kaehler manifold, and $\theta^1, \cdots, \theta^n$ a local field of unitary frames. The Chern form γ_k of M is given by

$$\gamma_k := \frac{(-1)^k}{(2\pi\sqrt{-1})^k k!}\sum \delta^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_k}\Omega^{\beta_1}_{\alpha_1}\wedge\cdots\wedge\Omega^{\beta_k}_{\alpha_k},$$

where $\Omega^{\alpha}_{\beta} := -\sum R^{\alpha}_{\beta\gamma\bar{\delta}}\theta^\gamma\wedge\bar{\theta}^\delta$. Let Λ be the operator of the interior product by the fundamental 2-form $\omega=\frac{\sqrt{-1}}{2}\sum\theta^\alpha\wedge\bar{\theta}^\alpha$. Then, after calculations, we obtain the following

$$(3.1)\quad \begin{aligned} \Lambda^3\gamma_1\wedge\gamma_2 = \frac{3}{64\pi^3}\{&\tau^3-8\tau|\rho|^2+\tau|R|^2+16\sum R_{\alpha\bar{\beta}}R_{\beta\bar{\gamma}}R_{\gamma\bar{\alpha}} \\ &-32\sum R_{\alpha\bar{\lambda}}R_{q\bar{p}}R^q{}_{p\lambda\bar{\alpha}}-64\sum R_{\alpha\bar{\gamma}}R^p{}_{q\gamma\bar{\lambda}}R^q{}_{p\lambda\bar{\alpha}}\}. \end{aligned}$$

In particular, if M is Einstein, then we have

$$(3.2)\quad \int_M \gamma_1\wedge\gamma_2\wedge\omega^{n-3}=\frac{3(n-1)!}{384\pi^3 n(n+1)}\int_M \tau\left(\tau^2+\frac{n+1}{n-1}A\right)dM,$$ where

$A := |R|^2-2\tau^2/n(n+1)$. The case $n=3$ in (3. 2) is given by

$$(3.3)\quad c_1c_2[M]-\frac{\tau^3\,\mathrm{vol}\,M}{2\cdot 384\pi^3}=\frac{1}{384\pi^3}\int_M \tau\left(|R|^2-\frac{\tau^2}{6}\right)dM,$$ where

$c_1c_2[M] := \int_M \gamma_1\wedge\gamma_2$.

In order to prove Theorem 2, we use the following

Lemma 2 ([4], [7]). *Let M be a $2n$-dimensional compact Kaehler manifold. Then the inequality $A\geqq 0$ holds and the equality holds if and only if M is of constant holomorphic sectional curvature.*

By (3. 2) and Lemma 2 we have

Lemma 3. *Let M be a $2n$-dimensional compact Einstein Kaehler manifold. If $\tau>0\,(\tau<0)$, then* $384\,\pi^3 n\,(n+1)\int_M \gamma_1\wedge\gamma_2\wedge\omega^{n-3}-3(n-1)\,!\,\tau^3$ vol $M\geqq 0$ ($\leqq 0$) *and the equality holds if and only if M is of constant holomorphic sectional curvature.*

Remark. For the case $n=3$ in Lemma 3, putting $a(M):=c_1c_2[M]/24$, we obtain a generalization of Proposition 2 in [7]. The arithmetic genus of M is defined by $a(M):=1-h^{1,0}+h^{2,0}-h^{3,0}$, where $h^{p,0}$ denotes the dimension of the space of holomorphic p-forms of M. If $\tau>0$, then M is algebraic (Kodaira-Spencer, see [5]), and consequently $a(M)=c_1c_2[M]/24$ (Riemann-Roch-Hirzebruch, see [5]).

Proof of Theorem 2. Let M be a 6-dimensional compact harmonic Kaehler manifold. Then the equality $4\gamma-2\beta=-45\tau\ddot{f}(0)$ holds ([11]). Making use of this equality and (2. 5), we obtain

$$\chi(M)=\frac{1}{2\cdot 384\pi^3}\int_M \tau\left(\frac{\tau^2}{3}-|R|^2\right)dM. \tag{3. 4}$$

From (3. 3) and (3. 4) we have

$$12\left\{a(M)-\frac{\tau^3\,\mathrm{vol}\,M}{48\cdot 384\pi^3}\right\}=\frac{\tau^3\,\mathrm{vol}\,M}{12\cdot 384\pi^3}-\chi(M). \tag{3. 5}$$

Now, from the above remark and (3. 5) we obtain the assertion (3) of Theorem 2. The assertion (2) is obvious.

Finally, we prove the assertion (1). Assume $\tau>0$. As M is Einstein, (i) M is simply connected ([6]), and (ii) $a(M)=1$ because of $h^{p,0}=0$ for $1\leqq p\leqq 3$ ([13]). By (i) and the harmonicity of M, for every point p in M, all geodesics starting from p are simply closed and of the same length, and consequently $\chi(M)=4$ ([1], [2]). By $a(M)=1$, $\chi(M)=4$ and (3. 5), we have

$$a(M)=\frac{\tau^3\,\mathrm{vol}\,M}{48\cdot 384\pi^3}.$$

Therefore, again by the above remark, M is of constant holomorphic sectional curvature.

4. Appendix. In this section we calculate the characteristic functions of compact symmetric spaces of rank one. Let M be an n-dimensional Riemannian manifold ($n\geqq 2$), and m a point of M. We denote by s

the geodesic distance from m to the point in a normal neighborhood of m. Let $(s, u) := (s, u_2, \cdots, u_n)$ be a system of geodesic polar coordinates at m. Then the Riemannian metric is given by the form (g_{ij}), $g_{1j} = \delta_{1j}$. Fix $s > 0$, and put $\gamma(t) := \exp_m(tu)$. We take linearly independent vectors $A_2, \cdots, A_n$ on the tangent space to M at m orthogonal to u such that $Y_i(t) := (\exp_m)_*(tA_i)$ $(i = 2, \cdots, n)$ are Jacobi fields along the geodesic γ and $Y_2(s), \cdots, Y_n(s)$ are orthonormal. Put $\theta(tu) := |Y_2(t) \wedge \cdots \wedge Y_n(t)| / t^{n-1} |A_2 \wedge \cdots \wedge A_n|$ and $g(s, u) := \det(g_{ij})$. Then we have the following equalities ([3]):

$$\Delta s = \frac{1}{\sqrt{g(s, u)}} \frac{\partial}{\partial s} \sqrt{g(s, u)} = \frac{\theta'}{\theta} + \frac{n-1}{s} = \sum_{i=2}^{n} \langle Y_i', Y_i \rangle (s).$$

Using the well-known Jacobi fields of compact symmetric spaces of rank one (see, for instance [2]) and the formula mentioned above, we obtain the following equalities:

1. The sphere S^n (with constant sectional curvature k):

$$f(\Omega) = 1 + 2(n-1)\, l \cot 2l, \quad \text{where } l := \sqrt{k}s/2.$$

2. The complex projective space PC^n (with constant holomorphic sectional curvature k):

$$f(\Omega) = 1 + (2n-1)\, l \cot l - l \tan l.$$

3. The quaternion projective space PQ^n (with maximal sectional curvature k):

$$f(\Omega) = 1 + (4n-1)\, l \cot l - 3l \tan l.$$

4. The Cayley projective plane PC_a^2 (with maximal sectional curvature k):

$$f(\Omega) = 1 + 15l \cot l - 7l \tan l.$$

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