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NUMERICAL INVESTIGATIONS OF C_k-NUMBERS

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Let n be a C_k-number, i. e. an integer such that $n>k>0$ and the congruence

$$a^{n-k} \equiv 0 \pmod{n}$$

is satisfied for any postitive integer a relatively prime to n.

In the previous note [2], we investigated the distribution of C_k-numbers for $1 \leqq k \leqq 60$ up to $n \leqq 4 \cdot 10^7$ and observed some distinguished features of it.

The present note is a report of our subsequent experiments on C_k-numbers. This time, by applying one more results due to W. Knödel as a sieve, we found an effective method of computing C_k-numbers in the definite range of k. In the sequel, we have executed for $1 \leqq k \leqq 60$ examining C_k-numbers up to the bound 10^9, with reflections on the results obtained.

All of the computation of our experiments was accomplished by making use of a computer HITAC 20 in the Department of Mathematics, Okayama University.

1. Method. In principle, the computing method of the present experiments is the same with the previous one, namely we have based on the following.

Theorem 1 (W. Knödel [1]). *A positive integer n is a C_k-number if and only if $n>k>0$ and $n-k$ is divisible by $\lambda(n)$, where $\lambda(n)$ is the Carmichael function.*

In the previous experiments, we attempted to investigate also the distribution of the sums of the Carmichael function $\lambda(n)$, so that we had to factorize a series of consecutive integers completely. Because of this affair, we were obliged to waste the long computing time.

In the present experiments, we have devoted ourselves to computing the C_k-numbers for each k with $1 \leqq k \leqq 60$ up to the bound N as much large as possible. To this end, we shall take advantage of the following theorem as a sieve.

Theorem 2 (W. Knödel [1]). *Let n be a C_k-number and let*

$$n = \prod_{i=0}^{\infty} p_i^{c_i} \quad \text{and} \quad k = \prod_{i=0}^{\infty} p_i^{k_i}$$

be the canonical factorizations of n *and* k *respectively, where* $p_0=2$.

Then, the following relations are satisfied.

$$(1) \quad c_i \leqq k_i + 1 \qquad \text{for } i \geqq 1,$$

$$(2) \quad c_0 \leqq k_0 + 2 \qquad \text{for } k_0 \geqq 1,$$

$$(3) \quad c_0 = k_0 \qquad \text{for } k_0 = 0.$$

Especially, when $(n, k) = 1$, *then* n *is an odd square-free integer.*

In actual processes, we may run on the scope of the Eratosthenes' sieve for each of a prime number p_i and its powers not exceeding the bound which is determined by the restriction $1 \leqq k \leqq 60$, in view of the conditions (1), (2) and (3) of Theorem 2.

2. Results and Considerations. Our results are summerized in Tables I and II below.

Table I is the table of $z(k, N)$ for $1 \leqq k \leqq 60$ and $N = 10^9$, where $z(k, N)$ designates the number of C_k-numbers not exceeding N.

In Table II, we have tabulated the quotient $Q(k, N) = kz(k, N)/z(1, N)$ for various bounds N. As is seen from the Table II, for each k, this quotient $Q(k, N)$ shows an inclination of converging to some stationary value depending on k.

In the following, we shall describe some of our observations on the dependence of this quotient $Q(k, N)$ on k.

Let p denote a prime number such that $p > k$. A C_k-number will be said to be of the first kind if it has the form kp, and otherwise of the second kind. Evidently, the prime numbers are C_1-numbers of the first kind and the C_1-numbers of the second kind are the Carmichael numbers. On the numerical evidence we are convinced that the C_k-numbers of the first kind may take the principal part in the distribution of C_k-numbers.

For examples, in the case of $k = 2$, an integer of the form $2p$ with p odd prime is always a C_2-number, since $\lambda(2p) = p - 1$ divides $2p - 2$, and therefore the asymptotic relation

$$2z(2, N) \sim z(1, N) \qquad (N \to \infty)$$

will probably be true. We believe that our experiments support this relation.

For the cases where $k = 3$ and $k = 4$, the aspects are the same as in the case of $k = 2$.

For the case of $k=5$,

$$\lambda(5p) = \text{LCM}(4, p-1) = \begin{cases} p-1 & \text{if } p \equiv 1 \pmod 4, \\ 2(p-1) & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Therefore, $\lambda(5p)$ divides $5p-5$ only for the case of $p \equiv 1 \pmod 4$. Since the prime numbers are distributed with equal density 1/2 in the arithmetic progressions $p \equiv 1 \pmod 4$ and $p \equiv 3 \pmod 4$, we may expect that the asymptotic relation

$$5z(5, N) \sim \frac{1}{2} z(1, N) \qquad (N \to \infty)$$

will hold true. This assertion can also be infered from our experiments.

In general, let p be a prime number such that $p \equiv r \pmod{\lambda(k)}$, where $\lambda(k)$ and r are naturally coprime. In this case, since

$$\lambda(kp) = \text{LCM}(\lambda(k), p-1) = \frac{\lambda(k)}{d}(p-1),$$

where $d=(\lambda(k), r-1)$, and an integer of the form kp is a C_k-number of the first kind, when and only when $\lambda(k)/d$ divides k. Hence, if t denotes the number of r, $0 \leqq r \leqq \lambda(k)$, such that k is divisible by $\lambda(k)/d$, then, we may consider that the asymptotic relation

$$kz(k, N)/z(1, N) \sim t/\phi(\lambda(k)) \qquad (N \to \infty)$$

takes place.

For example, in the case of $k=25$, we have $\lambda(k)=20$, $\phi(\lambda(k))=8$, and so:

r =	1	3	7	9	11	13	17	19
$r-1$ =	0	2	6	8	10	12	16	18
d =	20	2	2	4	2	4	4	2
$\lambda(k)/d$ =	1	10	10	5	10	5	5	10
$\frac{\lambda(k)}{d} \mid k$	○	×	×	○	×	○	○	×

Thus, we have $t=4$ and $t/\phi(\lambda(k))=4/8=0.5$. This value is approximately in conformity with our result $Q(25, 10^9)=0.5993$.

Table I. The number of C_k-numbers up to $N=10^9$.

k	$z(k, N)$	k	$z(k, N)$
1	50848180	31	249093
2	26357120	32	1934928
3	17957072	33	470869
4	13681572	34	456892
5	5540534	35	446554
6	9330155	36	1736112
7	4033324	37	140996
8	7113135	38	274943
9	6366010	39	804036
10	5764476	40	1572544
11	1317454	41	96153
12	4857438	42	1501470
13	1125712	43	122450
14	2100739	44	360474
15	1970139	45	706897
16	3705342	46	138287
17	437556	47	61818
18	3318133	48	1328168
19	525552	49	648940
20	3006019	50	1274299
21	2870101	51	157382
22	687628	52	615325
23	263761	53	50621
24	2535080	54	1188158
25	1218975	55	583506
26	1174347	56	576661
27	2267369	57	563483
28	1097846	58	185310
29	176921	59	39458
30	2057293	60	1081459

Table II. The Quotients $Q(k, N)$.

k \ N	10^7	10^8	10^9	$t/\phi(\lambda(k))$
1	1.0000	1.0000	1.0000	——
2	1.0493	1.0419	1.0366	1.0000
3	1.0806	1.0682	1.0594	1.0000
4	1.1042	1.0877	1.0762	1.0000
5	0.5617	0.5517	0.5448	0.5000
6	1.1407	1.1169	1.1009	1.0000
7	0.5760	0.5639	0.5552	0.5000
8	1.1678	1.1385	1.1191	1.0000
9	1.1738	1.1474	1.1267	1.0000
10	1.1883	1.1558	1.1336	1.0000
11	0.2995	0.2908	0.2850	0.2500
12	1.2161	1.1722	1.1463	1.0000
13	0.3031	0.2941	0.2878	0.2500
14	0.6144	0.5922	0.5783	0.5000
15	0.6239	0.5965	0.5811	0.5000
16	1.2448	1.1954	1.1659	1.0000
17	0.1564	0.1501	0.1462	0.1250
18	1.2626	1.2070	1.1746	1.0000
19	0.2101	0.2016	0.1963	0.1666
20	1.2954	1.2168	1.1823	1.0000
21	1.2719	1.2186	1.1853	1.0000
22	0.3231	0.3065	0.2975	0.2500
23	0.1304	0.1231	0.1193	0.1000
24	1.3116	1.2368	1.1965	1.0000
25	0.6491	0.6180	0.5993	0.5000
26	0.6482	0.6188	0.6004	0.5000
27	1.3056	1.2423	1.2039	1.0000
28	0.6713	0.6272	0.6045	0.5000
29	0.1116	0.1046	0.1009	0.0833
30	1.3399	1.2584	1.2137	1.0000

Table II. The Quotients $Q(k, N)$.

k \ N	10^7	10^8	10^9	$t/\phi(\lambda(k))$
31	0.1666	0.1572	0.1518	0.1250
32	1.3400	1.2614	1.2176	1.0000
33	0.3412	0.3179	0.3055	0.2500
34	0.3351	0.3163	0.3055	0.2500
35	0.3479	0.3213	0.3073	0.2500
36	1.3913	1.2818	1.2291	1.0000
37	0.1162	0.1071	0.1025	0.0833
38	0.2317	0.2144	0.2054	0.1666
39	0.6833	0.6407	0.6166	0.5000
40	1.3961	1.2901	1.2370	1.0000
41	0.0898	0.0812	0.0775	0.0625
42	1.3860	1.2920	1.2401	1.0000
43	0.1166	0.1081	0.1035	0.0833
44	0.3615	0.3279	0.3119	0.2500
45	0.7311	0.6589	0.6255	0.5000
46	0.1483	0.1322	0.1251	0.1000
47	0.0690	0.0607	0.0571	0.0454
48	1.4562	1.3179	1.2537	1.0000
49	0.6932	0.6504	0.6253	0.5000
50	1.3971	1.3053	1.2530	1.0000
51	0.1878	0.1674	0.1578	0.1250
52	0.7197	0.6589	0.6292	0.5000
53	0.0660	0.0560	0.0527	0.0416
54	1.4476	1.3227	1.2618	1.0000
55	0.7105	0.6593	0.6311	0.5000
56	0.7553	0.6720	0.6350	0.5000
57	0.7035	0.6586	0.6316	0.5000
58	0.2462	0.2223	0.2113	0.1666
59	0.0595	0.0495	0.0457	0.0357
60	1.5343	1.3543	1.2761	1.0000

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