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# NUMERICAL INVESTIGATIONS OF THE CARMICHAEL FUNCTION AND $C_k$-NUMBERS

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The Carmichael function $\lambda(n)$ introduced by R. D. Carmichael is, by definition, the smallest exponent $\lambda(n)$ such that for any positive integer $a$ relatively prime to a positive integer $n$, the congruece

$$a^{\lambda(n)} \equiv 1 \pmod{n}$$

is always satisfied.

This function is also characterized in terms of Euler's function $\phi(n)$ as follows:

$$\lambda(2^\alpha) = \phi(2^\alpha) \qquad \text{if } \alpha = 0, 1, 2,$$

$$\lambda(2^\alpha) = \frac{1}{2}\phi(2^\alpha) \qquad \text{if } \alpha > 2,$$

$$\lambda(p^\alpha) = \phi(p^\alpha) \qquad \text{if } p \text{ is an odd prime,}$$

$$\lambda(2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}) = \mathrm{LCM}\,(\lambda(2^\alpha),\ \lambda(p_1^{\alpha_1}),\ \lambda(p_2^{\alpha_2}), \cdots, \lambda(p_k^{\alpha_k})),$$

where 2, $p_1, p_2, \cdots, p_k$ are different prime numbers.

We put

$$L(N) = \sum_{n=1}^{N} \lambda(n).$$

In our experiments, we have pursued the study of the function $L(N)$ up to $N = 4 \cdot 10^7$ on a computer and have observed the behavior of this function.

In 1953, W. Knödel introduced a concept of $C_k$-numbers and discussed some properties of the $C_k$-numbers.

By his definition, an integer $n > 1$ is a $C_k$-number if $n > k > 0$ and the congruence

$$a^{n-k} \equiv 1 \pmod{n}$$

is satisfied for any positive integer $a$ relatively prime to $n$. Particularly, a $C_1$-number is a prime or a Carmichael number and a $C_3$-number is Morrow's $D$-number.

Let $z(k, N)$ be the number of $C_k$-numbers not exceeding $N$. In his paper [2], W. Knödel asserted that the upper estimation of $z(k, N)$ is

$$z(k, N) = O(kN \cdot 2^{-\sqrt{\log N}} + k^6),$$

and the lower estimation is

$$z(k,\ N) = \Omega(N/k \log N).$$

Moreover, he conjectured that the real order of magnitude would probably be

$$z(k,\ N) = O(kN^h) \quad \text{for some} \quad h < 1.$$

In our experiments, we have tried to support numerically his conjecture on a computer. However, we have observed that the distribution of $C_k$-numbers has slightly more complicated features.

All of computation of our experiments was done by making use of a computer HITAC 20 in the Department of Mathematics, Okayama University and a computer ACOS 700S in the Computer Ceter, Okayama University.

**1. Method.** The following Proposition gives a principle of computing $C_k$-numbers.

**Proposition** (W. Knödel). *A positive integer $n$ is a $C_k$-number if and only if $n > k > 0$ and $n - k$ is divisible by $\lambda(n)$.*

For computation of the Carmichael function $\lambda(n)$, we have adopted a method that can be described in terms of Euler's totient function.

In order to make factorizations of consecutive integers $n$, we have used the method of Eratosthenes' sieve with area of 20000 words for each of the three parameters, i. e. the number $n$, the exponent $\alpha$ of a prime factor and the value $\lambda$ of the Carmichael function.

Firstly, for each of a prime number $p$ and powers of $p$, we run on the area of the sieve and we count the exponent $\alpha$ of $p$. And then, we compute the value of the parameter $\lambda$ for $n$ having positive exponent $\alpha$ by the following successive process:

$$\text{Initial set}: \lambda = 1,$$
$$\text{Iteration}\ : \lambda = \lambda \cdot \lambda(p^\alpha)/\text{GCD}(\lambda,\ \lambda(p^\alpha)),$$

where GCD is computed by the so-called Euclidean algorithm.

After obtaining values of $\lambda(n)$, we now accumulate $\lambda(n)$ to $L(N)$ the sum of $\lambda(n)$ and test the divisibility of $n-k$ by $\lambda(n)$ for $1 \leqq k \leqq 60$.

**2. Results and Observations.** In the Table I. we have tabulated the sums $L(N)$ of the Carmichael function $\lambda(n)$ up to $N$ and the logarithmic ratios $R(N) = \log L(N)/\log N$. It seems likely that the column of the logarithmic ratios $R(N)$ in the Table I shows an indication of converging to some definite value.

In the Table II, we have tabulated the value of $z(k, N)$ for $1 \leqq k \leqq 60$ and $N = 4 \cdot 10^7$. At a glance, the array of values of $z(k, N)$ is likely unsteady. However, we have observed the fact that the quotients $kz(k, N)/z(1, N)$ show somewhat tendancy.

In the Table III, we have tabulated the quotients $Q(k, N) = kz(k, N)/z(1, N)$. If we are permitted to guess boldly, we may say that as $N$ is incresing for a fixed $k$, the quotient $Q(k, N)$ tends to a limit which is probably of the simple form, say, $1/2^i$ for some integer $i \geqq 0$ depending on $k$. But, in order to give a certain answer, we think that we need to develop a new method effective up to a fairly large number $N$.

**Table I. Sums of the Carmichael function.**

| $N$ | $L(N)$ | $R(N)$ |
|---|---|---|
| 10 | 30 | 1.477121 |
| $10^2$ | 2148 | 1.669626 |
| $10^3$ | 170830 | 1.744188 |
| $10^4$ | 13777264 | 1.784791 |
| $10^5$ | 1163833056 | 1.813178 |
| $10^6$ | 101550794084 | 1.834447 |
| $10^7$ | 9058117030782 | 1.851006 |
| $2 \cdot 10^7$ | 35122111222286 | 1.855298 |
| $3 \cdot 10^7$ | 77673469605446 | 1.857681 |
| $4 \cdot 10^7$ | 136353639402164 | 1.859321 |

**Table II. The number of $C_k$-numbers up to $N = 4\cdot10^7$**

| $k$ | $z(k, N)$ | $k$ | $z(k, N)$ |
|---|---|---|---|
| 1 | 2433832 | 31 | 12589 |
| 2 | 1270982 | 32 | 97837 |
| 3 | 870051 | 33 | 23979 |
| 4 | 665275 | 34 | 23107 |
| 5 | 270226 | 35 | 22923 |
| 6 | 456356 | 36 | 88914 |
| 7 | 197616 | 37 | 7185 |
| 8 | 349443 | 38 | 14064 |
| 9 | 313141 | 39 | 40849 |
| 10 | 284004 | 40 | 80424 |
| 11 | 65000 | 41 | 4987 |
| 12 | 240664 | 42 | 76637 |
| 13 | 55652 | 43 | 6285 |
| 14 | 104261 | 44 | 18683 |
| 15 | 98202 | 45 | 36822 |
| 16 | 184269 | 46 | 7258 |
| 17 | 21841 | 47 | 3277 |
| 18 | 165606 | 48 | 68942 |
| 19 | 26207 | 49 | 32966 |
| 20 | 150367 | 50 | 64987 |
| 21 | 143302 | 51 | 8264 |
| 22 | 34490 | 52 | 31701 |
| 23 | 13300 | 53 | 2669 |
| 24 | 127810 | 54 | 61336 |
| 25 | 61160 | 55 | 29903 |
| 26 | 58860 | 56 | 30312 |
| 27 | 113851 | 57 | 28753 |
| 28 | 55654 | 58 | 9603 |
| 29 | 8959 | 59 | 2160 |
| 30 | 104162 | 60 | 57082 |

**Table III. The quotients $Q(k, N)$.**

| $k$ \ $N$ | $10^6$ | $10^7$ | $2 \cdot 10^7$ | $4 \cdot 10^7$ |
|---|---|---|---|---|
| 1 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 2 | 1.0604 | 1.0493 | 1.0464 | 1.0444 |
| 3 | 1.0994 | 1.0806 | 1.0760 | 1.0724 |
| 4 | 1.1320 | 1.1042 | 1.0987 | 1.0933 |
| 5 | 0.5783 | 0.5617 | 0.5581 | 0.5551 |
| 6 | 1.1822 | 1.1407 | 1.1326 | 1.1250 |
| 7 | 0.5954 | 0.5760 | 0.5721 | 0.5689 |
| 8 | 1.2213 | 1.1678 | 1.1570 | 1.1486 |
| 9 | 1.2301 | 1.1738 | 1.1672 | 1.1579 |
| 10 | 1.2455 | 1.1883 | 1.1771 | 1.1669 |
| 11 | 0.3140 | 0.2995 | 0.2962 | 0.2937 |
| 12 | 1.3058 | 1.2161 | 1.1993 | 1.1865 |
| 13 | 0.3199 | 0.3031 | 0.3005 | 0.2972 |
| 14 | 0.6532 | 0.6144 | 0.6059 | 0.5997 |
| 15 | 0.6777 | 0.6239 | 0.6132 | 0.6052 |
| 16 | 1.3447 | 1.2448 | 1.2262 | 1.2113 |
| 17 | 0.1716 | 0.1564 | 0.1542 | 0.1525 |
| 18 | 1.3775 | 1.2626 | 1.2415 | 1.2247 |
| 19 | 0.2266 | 0.2101 | 0.2071 | 0.2045 |
| 20 | 1.3972 | 1.2954 | 1.2526 | 1.2356 |
| 21 | 1.3644 | 1.2719 | 1.2524 | 1.2364 |
| 22 | 0.3562 | 0.3231 | 0.3169 | 0.3117 |
| 23 | 0.1467 | 0.1304 | 0.1278 | 0.1256 |
| 24 | 1.4924 | 1.3116 | 1.2839 | 1.2603 |
| 25 | 0.7037 | 0.6491 | 0.6373 | 0.6282 |
| 26 | 0.7074 | 0.6482 | 0.6375 | 0.6287 |
| 27 | 1.4290 | 1.3056 | 1.2824 | 1.2630 |
| 28 | 0.7661 | 0.6713 | 0.6543 | 0.6402 |
| 29 | 0.1310 | 0.1116 | 0.1094 | 0.1067 |
| 30 | 1.5175 | 1.3399 | 1.3083 | 1.2839 |

**Table III. The Quotients $Q(k, N)$.**

| $k$ \ $N$ | $10^6$ | $10^7$ | $2\cdot 10^7$ | $4\cdot 10^7$ |
|---|---|---|---|---|
| 31 | 0.1890 | 0.1666 | 0.1629 | 0.1603 |
| 32 | 1.4745 | 1.3400 | 1.3111 | 1.2863 |
| 33 | 0.3999 | 0.3412 | 0.3329 | 0.3251 |
| 34 | 0.3787 | 0.3351 | 0.3284 | 0.3227 |
| 35 | 0.4015 | 0.3479 | 0.3374 | 0.3296 |
| 36 | 1.6693 | 1.3913 | 1.3479 | 1.3151 |
| 37 | 0.1380 | 0.1162 | 0.1121 | 0.1092 |
| 38 | 0.2777 | 0.2317 | 0.2247 | 0.2195 |
| 39 | 0.7716 | 0.6833 | 0.6668 | 0.6545 |
| 40 | 1.6485 | 1.3961 | 1.3547 | 1.3217 |
| 41 | 0.1153 | 0.0898 | 0.0864 | 0.0840 |
| 42 | 1.6021 | 1.3860 | 1.3508 | 1.3225 |
| 43 | 0.1341 | 0.1166 | 0.1137 | 0.1110 |
| 44 | 0.4599 | 0.3615 | 0.3482 | 0.3377 |
| 45 | 0.9075 | 0.7311 | 0.7022 | 0.6808 |
| 46 | 0.2038 | 0.1483 | 0.1423 | 0.1371 |
| 47 | 0.0939 | 0.0690 | 0.0654 | 0.0632 |
| 48 | 1.8389 | 1.4562 | 1.4003 | 1.3596 |
| 49 | 0.7717 | 0.6932 | 0.6762 | 0.6636 |
| 50 | 1.5749 | 1.3971 | 1.3628 | 1.3350 |
| 51 | 0.2584 | 0.1878 | 0.1802 | 0.1731 |
| 52 | 0.8560 | 0.7197 | 0.6948 | 0.6773 |
| 53 | 0.1025 | 0.0660 | 0.0609 | 0.0581 |
| 54 | 1.7621 | 1.4476 | 1.3984 | 1.3608 |
| 55 | 0.8130 | 0.7105 | 0.6911 | 0.6757 |
| 56 | 0.9761 | 0.7553 | 0.7234 | 0.6974 |
| 57 | 0.8048 | 0.7035 | 0.6882 | 0.6733 |
| 58 | 0.3175 | 0.2462 | 0.2359 | 0.2284 |
| 59 | 0.1878 | 0.0595 | 0.0553 | 0.0523 |
| 60 | 2.0175 | 1.5343 | 1.4625 | 1.4072 |

## REFERENCES

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