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ERGODIC THEOREMS FOR d-PARAMETER SEMIGROUPS OF DUNFORD-SCHWARTZ OPERATORS

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1. Introduction. Let $\Gamma=\{T(t_1, \cdots, t_d);\ t_1, \cdots, t_d>0\}$ be a strongly continuous d-parameter semigroup of Dunford-Schwartz operators on $L_1(\Omega)=L_1(\Omega, \mathscr{F}, \mu)$, where $(\Omega, \mathscr{F}, \mu)$ is a σ-finite measure space. In this paper Γ will be extended to a semigroup of linear operators on the class $L_1(\Omega)+L_\infty(\Omega)$ of all functions f of the form $f=g+h$, with $g\in L_1(\Omega)$ and $h\in L_\infty(\Omega)$, so that $\|T(t_1, \cdots, t_d)\|_p\leq 1$ for every $1\leq p\leq\infty$ and also so that $\lim_n T(t_1, \cdots, t_d)f_n=T(t_1, \cdots, t_d)f$ almost everywhere on Ω whenever $f_n\in L_\infty(\Omega)$, $\sup_n\|f_n\|_\infty<\infty$ and $\lim_n f_n=f$ almost everywhere on Ω. Then for every $f\in L_1(\Omega)+L_\infty(\Omega)$ the averages

$$A(\alpha_1, \cdots, \alpha_d)f=\frac{1}{\alpha_1\cdots\alpha_d}\int_0^{\alpha_1}\cdots\int_0^{\alpha_d}T(t_1, \cdots, t_d)f\,dt_1\cdots dt_d$$

are well-defined, and now it would be interesting to ask the following questions: For what functions f does the almost everywhere convergence of the averages $A(\alpha_1, \cdots, \alpha_d)f$ hold as $\alpha_1\longrightarrow 0, \cdots, \alpha_d\longrightarrow 0$ independently? For what functions f does the almost everywhere convergence of the averages $A(\alpha_1, \cdots, \alpha_d)f$ hold as $\alpha_1\longrightarrow\infty, \cdots, \alpha_d\longrightarrow\infty$ independently?

It will be proved below that if $f\in L_1(\Omega)+L_\infty(\Omega)$ satisfies

$$\int_{\{|f|>t\}}\frac{|f|}{t}\left[\log\frac{|f|}{t}\right]^{d-1}d\mu<\infty$$

for every $t>0$, then the averages $A(\alpha_1, \cdots, \alpha_d)f$ converge almost everywhere on Ω as $\alpha_1\longrightarrow 0, \cdots, \alpha_d\longrightarrow 0$ independently, and also the averages $A(\alpha_1, \cdots, \alpha_d)f$ converge almost everywhere on Ω as $\alpha_1\longrightarrow\infty, \cdots, \alpha_d\longrightarrow\infty$ independently. This may be considered to be an extension of Terrell's local ergodic theorem [10] and Dunford-Schwartz's ergodic theorem [2].

The method of proof chiefly depends upon a weak type maximal inequality similar to Fava's [4].

2. Preliminaries. Let $(\Omega, \mathscr{F}, \mu)$ be a σ-finite measure space and let $L_p(\Omega)=L_p(\Omega, \mathscr{F}, \mu)$, $1\leq p\leq\infty$, be the usual Banach spaces of real or complex functions on $(\Omega, \mathscr{F}, \mu)$. A *Dunford-Schwartz operator* T on $L_1(\Omega)$

is a linear contraction on $L_1(\Omega)$ (i.e. $\|T\|_1 \leq 1$) such that for every $f \in L_1(\Omega) \cap L_\infty(\Omega)$

$$\|Tf\|_\infty \leq \|f\|_\infty.$$

It is well-known that a Dunford-Schwartz operator T on $L_1(\Omega)$ satisfies

$$\|Tf\|_p \leq \|f\|_p$$

for all $f \in L_1(\Omega) \cap L_p(\Omega)$, with $1 < p < \infty$. By this, T can be uniquely extended to a linear contraction on each $L_p(\Omega)$, with $1 < p < \infty$. Furthermore it can be extended to a linear contraction on $L_\infty(\Omega)$ as follows. If $0 \leq f \in L_\infty(\Omega)$, choose $f_n \in L_1(\Omega)$ so that $0 \leq f_n \leq f_{n+1} \leq f$ and $\lim_n f_n = f$ almost everywhere (a. e.) on Ω. Then for $n > m$ we have

$$|Tf_n - Tf_m| \leq \tau(f_n - f_m) \leq (\lim_k \tau f_k) - \tau f_m \quad \text{a. e. on } \Omega$$

where τ denotes the linear modulus of T in the sense of Chacon-Krengel [1]. (Thus τ is a positive Dunford-Schwartz operator on $L_1(\Omega)$ such that

$$\tau g = \sup \{|Th| : h \in L_1(\Omega),\ |h| \leq g \ \text{a. e. on } \Omega\}$$

for any $0 \leq g \in L_1(\Omega)$.) On the other hand, if $0 \leq u \in L_1(\Omega) \cap L_\infty(\Omega)$ and $0 < u$ a. e. on Ω, then it may be readily seen that $0 \leq \tau^* u \in L_1(\Omega)$ and $\|\tau^* u\|_1 \leq \|u\|_1$, where τ^* denotes the adjoint operator of τ, acting on $L_\infty(\Omega) = L_1(\Omega)^*$. Thus, putting

$$g_m = (\lim_k \tau f_k) - \tau f_m \quad \text{a. e. on } \Omega,$$

we have, by Lebesgue's dominated convergence theorem,

$$\int g_m u \, d\mu = \int (\lim_k f_k) \, \tau^* u \, d\mu - \int f_m \, \tau^* u \, d\mu \longrightarrow 0$$

as $m \longrightarrow \infty$. Since $u > 0$ a. e. on Ω and $g_m \geq g_{m+1} \geq 0$ a. e. on Ω, this proves that $\lim_m g_m = 0$ a. e. on Ω, and hence for almost all $\omega \in \Omega$ the sequence $Tf_n(\omega)$, $n = 1, 2, \cdots$, is a Cauchy sequence. Therefore it is possible to define

$$Tf(\omega) = \lim_n Tf_n(\omega) \quad \text{a. e. on } \Omega.$$

It is now a routine matter to check that this definition of Tf does not depend upon the particular choice of such a sequence (f_n) in $L_1(\Omega)$, and so by linearity T can be extended to a linear operator on $L_\infty(\Omega)$. From the definition of T on $L_\infty(\Omega)$, it follows that $\|T\|_\infty \leq 1$ and that if $f_n \in L_\infty(\Omega)$, $n = 1, 2, \cdots$, is a sequence satisfying $\sup_n \|f_n\|_\infty < \infty$ and $\lim_n f_n = f$ a. e. on Ω for some $f \in L_\infty(\Omega)$, then

$$Tf=\lim_n Tf_n \quad \text{a. e. on } \Omega.$$

The above discussion ensures that we may and will assume, throughout this paper, that a Dunford-Schwartz operator T is a linear operator on the class $L_1(\Omega)+L_\infty(\Omega)$ such that $\|T\|_p \leq 1$ on each $L_p(\Omega)$ with $1 \leq p \leq \infty$ and also such that

$$Tf=\lim_n Tf_n \quad \text{a. e. on } \Omega$$

whenever $f_n \in L_\infty(\Omega)$, $\sup\{\|f_n\|_\infty : n \geq 1\} < \infty$ and $f=\lim_n f_n$ a. e. on Ω.

Let us now consider a d-parameter semigroup $\Gamma=\{T(t_1, \cdots, t_d)\,;\, t_1, \cdots, t_d>0\}$ of Dunford-Schwartz operators on $L_1(\Omega)+L_\infty(\Omega)$, $d \geq 1$ being a fixed integer. Thus each $T(t_1, \cdots, t_d)$ is a Dunford-Schwartz operator on $L_1(\Omega)+L_\infty(\Omega)$, and Γ satisfies

$$T(t_1, \cdots, t_d)T(s_1, \cdots, s_d)=T(t_1+s_1, \cdots, t_d+s_d).$$

Throughout this paper we shall assume that Γ is strongly continuous with respect to the norm topology of $L_1(\Omega)$, i. e. for each $f \in L_1(\Omega)$ the function $T(t_1, \cdots, t_d)f$ of $(t_1, \cdots, t_d) \in R_+^d$, where $R_+^d=\{(t_1, \cdots, t_d) : t_1, \cdots, t_d>0\}$, is continuous with respect to the norm topology of $L_1(\Omega)$. It then follows from an approximation argument that Γ is strongly continuous with respect to the norm topology of each $L_p(\Omega)$ with $1 \leq p < \infty$, and that for each $f \in L_p(\Omega)$ with $1 \leq p < \infty$ there exists a scalar function $g(t_1, \cdots, t_d, \omega)$, defined on $R_+^d \times \Omega$ and measurable with respect to the product of the Lebesgue measurable subsets of R_+^d and $\mathscr{F}$, such that for each fixed $(t_1, \cdots, t_d) \in R_+^d$, $g(t_1, \cdots, t_d, \omega)$ as a function of $\omega \in \Omega$ belongs to the equivalence class of $T(t_1, \cdots, t_d)f \in L_p(\Omega)$. Therefore, in the sequel, $g(t_1, \cdots, t_d, \omega)$ will be denoted by $T(t_1, \cdots, t_d)f(\omega)$. It then follows from Fubini's theorem that there exists a μ-null set $E(f)$, dependent on f but independent of $(t_1, \cdots, t_d)$, such that for each fixed $\omega \notin E(f)$, $T(t_1, \cdots, t_d)f(\omega)$ as a function of $(t_1, \cdots, t_d) \in R_+^d$ is Lebesgue integrable over every finite interval $(\alpha_1, \beta_1) \times \cdots \times (\alpha_d, \beta_d) \subset R_+^d$ with respect to the Lebesgue measure, and the integral

$$\int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)f(\omega)\, dt_1 \cdots dt_d \quad (\omega \notin E(f))$$

as a function of $\omega \in \Omega$ belongs to the equivalence class of the Bochner integral

$$\int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)f\, dt_1 \cdots dt_d \quad (\in L_p(\Omega)).$$

Next we will observe that a similar situation holds for $f \in L_\infty(\Omega)$. In fact, let (f_n) be a sequence in $L_1(\Omega)$ such that $|f_n| \leq |f|$ and $\lim_n f_n=f$ a. e.

on Ω. Then for every $(t_1, \cdots, t_d) \in R_+^d$

$$T(t_1, \cdots, t_d)f = \lim_n T(t_1, \cdots, t_d)f_n \quad \text{a. e. on } \Omega,$$

and hence by Fubini's theorem we may define

$$g(t_1, \cdots, t_d, \omega) = \lim_n T(t_1, \cdots, t_d)f_n(\omega)$$

for almost all $(t_1, \cdots, t_d, \omega) \in R_+^d \times \Omega$ with respect to the product of the Lebesgue measure and μ. Since, for each fixed $(t_1, \cdots, t_d) \in R_+^d$, $g(t_1, \cdots, t_d, \omega)$ as a function of $\omega \in \Omega$ belongs to the equivalence class of $T(t_1, \cdots, t_d)f \in L_\infty(\Omega)$, $g(t_1, \cdots, t_d, \omega)$ will be again denoted by $T(t_1, \cdots, t_d)f(\omega)$. It then follows from Fubini's theorem that there exists a μ-null set $E(f)$, dependent on f but independent of $(t_1, \cdots, t_d)$, such that for each fixed $\omega \notin E(f)$, $T(t_1, \cdots, t_d)f(\omega)$ as a function of $(t_1, \cdots, t_d) \in R_+^d$ is Lebesgue integrable over every finite interval $(\alpha_1, \beta_1) \times \cdots \times (\alpha_d, \beta_d) \subset R_+^d$, and the integral

$$\int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)f(\omega)\, dt_1 \cdots dt_d \quad (\omega \notin E(f))$$

as a function of $\omega \in \Omega$ belongs to $L_\infty(\Omega)$ and satisfies

$$\begin{aligned} &< \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)f(\omega)\, dt_1 \cdots dt_d,\ u(\omega) > \\ &= \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} < f,\ T(t_1, \cdots, t_d)^* u > dt_1 \cdots dt_d \end{aligned}$$

for all $u \in L_1(\Omega) \cap L_\infty(\Omega)$ (where we let $< f,\ u > = \int_\Omega fu\, d\mu$) and hence for all $u \in L_1(\Omega)$, because the adjoint semigroup $\Gamma^* = \{T(t_1, \cdots, t_d)^*\,;\ t_1, \cdots, t_d > 0\}$ may be regarded as a semigroup of Dunford-Schwartz operators on $L_1(\Omega) + L_\infty(\Omega)$ which is strongly continuous with respect to the norm topology of $L_1(\Omega)$.

Now let f be in the class $L_1(\Omega) + L_\infty(\Omega)$ and write $f = g + h$ with $g \in L_1(\Omega)$ and $h \in L_\infty(\Omega)$. Then we may define the integral

$$\int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)f\, dt_1 \cdots dt_d \quad (\in L_1(\Omega) + L_\infty(\Omega))$$

over the finite interval $(\alpha_1, \beta_1) \times \cdots \times (\alpha_d, \beta_d) \subset R_+^d$ to be the function

$$\begin{aligned} &\left(\int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)f\, dt_1 \cdots dt_d\right)(\omega) \\ &= \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)g(\omega)\, dt_1 \cdots dt_d \\ &+ \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d)h(\omega)\, dt_1 \cdots dt_d \quad \text{a. e. on } \Omega. \end{aligned}$$

It is clear that this definition of the integral does not depend upon the particular choice of such a decomposition $f=g+h$, and we have the relation

$$< \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} T(t_1, \cdots, t_d) f \, dt_1 \cdots dt_d, \ u >$$
$$= \int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_d}^{\beta_d} < T(t_1, \cdots, t_d) f, \ u > dt_1 \cdots dt_d$$

for all $u \in L_1(\Omega) \cap L_\infty(\Omega)$.

3. Maximal operators and inequalities. We will call an operator M, which maps functions in $L_1(\Omega)+L_\infty(\Omega)$ into measurable functions on $(\Omega, \mathscr{F})$, a *maximal operator* if it satisfies:

(a) $|M(f+g)| \leq |Mf| + |Mg|$ and $|M(cf)| = |c| \, |Mf|$ a. e. on Ω where c is a constant;

(b) There exists a constant $A>0$ such that for every $f \in L_1(\Omega)+L_\infty(\Omega)$ and all $\lambda > 0$

$$\|Mf\|_\infty \leq A\|f\|_\infty \text{ and } \mu\{|Mf| > \lambda\} \leq \frac{A}{\lambda} \|f\|_1.$$

Lemma 1. *If M is a maximal operator on $L_1(\Omega)+L_\infty(\Omega)$ then for every $f \in L_1(\Omega)+L_\infty(\Omega)$ and all $t>0$*

$$\mu\{|Mf| > (A+1)t\} \leq \frac{A}{t} \int_{\{|f|>t\}} |f| \, d\mu.$$

Proof. Putting $f^t(\omega) = f(\omega) 1_{\{|f|>t\}}(\omega)$ and $f_t(\omega) = f(\omega) - f^t(\omega)$, we have $\|f_t\|_\infty \leq t$ and

$$|Mf| \leq |M(f^t)| + |M(f_t)| \leq |M(f^t)| + At.$$

Thus $\{|Mf| > (A+1)t\} \subset \{|M(f^t)| > t\}$ and consequently we have

$$\mu\{|Mf| > (A+1)t\} \leq \mu\{|M(f^t)| > t\}$$
$$\leq \frac{A}{t} \int_{\{|f|>t\}} |f| \, d\mu,$$

which completes the proof.

Corollary. *If M is a maximal operator on $L_1(\Omega)+L_\infty(\Omega)$ then for every $f \in L_1(\Omega)+L_\infty(\Omega)$*

$$\int |Mf|^p \, d\mu \leq \frac{pA(A+1)^p}{p-1} \int |f|^p \, d\mu \quad (1<p<\infty)$$

and

$$\int |Mf|\,d\mu \le (A+1)\,[\mu(\Omega)+A\int_{\{|f|>1\}} |f| \log |f|\,d\mu].$$

Proof. If $1<p<\infty$ then, by Fubini's theorem,

$$\int |Mf|^p\,d\mu = p\int_0^\infty r^{p-1}\mu\{|Mf|>r\}\,dr$$
$$\le p\int_0^\infty dr\left[r^{p-1}\frac{A(A+1)}{r}\int_{\{|f|>\frac{r}{A+1}\}} |f|\,d\mu\right]$$
$$= pA(A+1)\int_\Omega d\mu(\omega)[|f(\omega)|\int_0^{(A+1)|f(\omega)|} r^{p-2}\,dr]$$
$$= \frac{pA(A+1)^p}{p-1}\int_\Omega |f(\omega)|^p\,d\mu(\omega),$$

and if $p=1$ then, again by Fubini's theorem,

$$\int |Mf|\,d\mu = \int_0^\infty \mu\{|Mf|>r\}\,dr$$
$$\le (A+1)\,\mu(\Omega)+\int_{A+1}^\infty \mu\{|Mf|>r\}\,dr$$
$$\le (A+1)\,\mu(\Omega)+\int_{A+1}^\infty dr\left[\frac{A(A+1)}{r}\int_{\{|f|>\frac{r}{A+1}\}} |f|\,d\mu\right]$$
$$= (A+1)\,\mu(\Omega)+A(A+1)\int_{\{|f|>1\}} d\mu(\omega)\left[|f(\omega)|\int_1^{|f(\omega)|}\frac{1}{r}dr\right]$$
$$= (A+1)\,\mu(\Omega)+A(A+1)\int_{\{|f|>1\}} |f(\omega)| \log |f(\omega)|\,d\mu(\omega)$$

Hence the proof is completed. (This argument is standard.)

For each $n\ge 0$, let $R_n(\Omega)$ be the class of all functions f in $L_1(\Omega)+L_\infty(\Omega)$ such that

$$\int_{\{|f|>t\}} \frac{|f|}{t}\left[\log\frac{|f|}{t}\right]^n d\mu < \infty$$

for all $t>0$, and let $L(\Omega)[\log^+ L(\Omega)]^n$ be the class of all functions f in $L_1(\Omega)+L_\infty(\Omega)$ such that

$$\int_{\{|f|>1\}} |f|\ [\log|f|]^n\,d\mu < \infty.$$

The classes $R_n(\Omega)$, $n\ge 0$, originally introduced by Fava [4] in order to obtain a weak type inequality for a product of maximal operators, have the following properties:

(i) For each $n\geq 0$, $R_n(\Omega)\subset L(\Omega)[\log^+ L(\Omega)]^n$ and both classes coincide if and only if $\mu(\Omega)<\infty$.

(ii) $L_1(\Omega)\subset R_0(\Omega)$ and both classes coincide if and only if $\mu(\Omega)<\infty$.

(iii) For each $n\geq 0$, $R_{n+1}(\Omega)\subset R_n(\Omega)$ and both classes coincide if and only if there exists a constant $\delta>0$ such that $E\in\mathscr{F}$ and $\mu(E)>0$ implies $\mu(E)>\delta$.

(iv) For each $n\geq 0$, $R_n(\Omega)$ is a linear manifold of $L_1(\Omega)+L_\infty(\Omega)$.

(v) For each $n\geq 0$, $R_n(\Omega)$ includes the linear manifold generated by $\bigcup_{1<p<\infty} L_p(\Omega)$, and both manifolds coincide if and only if $\mu(\Omega)<\infty$ and there exists a constant $\delta>0$ such that $E\in\mathscr{F}$ and $\mu(E)>0$ implies $\mu(E)>\delta$.

Some of the above properties are found in [4] and the others may be directly proved, and hence we omit the details.

The following maximal theorem is a key lemma to prove individual ergodic theorems for d-parameter semigroups $\Gamma=\{T(t_1,\cdots,t_d);\ t_1,\cdots,t_d>0\}$ of Dunford-Schwartz operators on $L_1(\Omega)+L_\infty(\Omega)$.

Theorem 1. *Let M be a maximal operator on $L_1(\Omega)+L_\infty(\Omega)$, and let $A>0$ be the constant relating to M as in the definition of a maximal operator. Then for each $n\geq 0$ there corresponds a constant $B_n=B(n, A)>0$ so that for every $f\in R_{n+1}(\Omega)$ and all $t>0$*

$$\int_{\{|Mf|>t\}} \frac{|Mf|}{t}\left[\log\frac{|Mf|}{t}\right]^n d\mu\leq \int_{\{B_n|f|>t\}} \frac{B_n|f|}{t}\left[\log\frac{B_n|f|}{t}\right]^{n+1} d\mu.$$

Consequently $f\in R_{n+1}(\Omega)$ implies $Mf\in R_n(\Omega)$.

Proof. Fix any $a>1$. Then for $f\in R_{n+1}(\Omega)$ and $t>0$, putting $g=f/t$, we have by Fubini's theorem

$$\begin{aligned}
\int_{\{|Mf|>at\}} \frac{|Mf|}{t}\left[\log\frac{|Mf|}{t}\right]^n d\mu &= \int_{\{|Mg|>a\}} |Mg|[\log|Mg|]^n\, d\mu\\
&=\int_1^\infty \mu(\{|Mg|>a\}\cap\{|Mg|>r\})([\log r]^n+n[\log r]^{n-1})\,dr\\
&=\int_1^a \mu\{|Mg|>a\}([\log r]^n+n[\log r]^{n-1})\,dr\\
&\quad+\int_a^\infty \mu\{|Mg|>r\}([\log r]^n+n[\log r]^{n-1})\,dr\\
&=I+II.
\end{aligned}$$

Since Lemma 1 implies that

$$\mu\{|Mg|>r\} \le \frac{A(A+1)}{r} \int_{\{(A+1)|g|>r\}} |g|\, d\mu \qquad (r>0),$$

it follows that

$$I \le \frac{A}{a} \int_{\{(A+1)|g|>a\}} (A+1)|g|\, d\mu \times \int_1^a ([\log r]^n + n[\log r]^{n-1})\, dr$$
$$\le I(A) \int_{\{(A+1)|g|>a\}} (A+1)|g|\, [\log (A+1)|g|]^{n+1}\, d\mu,$$

where

$$I(A) = \frac{A}{a}[\log a]^{-(n+1)} \int_1^a ([\log r]^n + n[\log r]^{n-1})\, dr.$$

Further, since $a>1$ and $n\ge 0$, it follows that

$$II \le \int_a^\infty \Big[\Big(\frac{A(A+1)}{r} \int_{\{(A+1)|g|>r\}} |g|\, d\mu\Big)([\log r]^n + n[\log r]^{n-1})\Big]\, dr$$
$$= \int_{\{(A+1)|g|>a\}} d\mu(\omega) \Big[A(A+1)|g(\omega)| \int_a^{(A+1)|g(\omega)|} \Big(\frac{1}{r}[\log r]^n + \frac{n}{r}[\log r]^{n-1}\Big)\, dr\Big]$$
$$\le \int_{\{(A+1)|g|>a\}} d\mu(\omega) \Big[A(A+1)|g(\omega)| ([\log (A+1)|g(\omega)|]^{n+1}$$
$$+ [\log (A+1)|g(\omega)|]^n)\Big]$$
$$\le A(1+[\log a]^{-1}) \int_{\{(A+1)|g|>a\}} (A+1)|g(\omega)|\, [\log (A+1)|g(\omega)|]^{n+1}\, d\mu(\omega).$$

Thus, letting $B_n = \Big[I(A) + A(1+[\log a]^{-1}) + 1\Big](A+1)$, we get

$$\int_{\{|Mf|>at\}} \frac{|Mf|}{t} [\log \frac{|Mf|}{t}]^n\, d\mu \le \int_{\{B_n|f|>at\}} \frac{B_n|f|}{t} [\log \frac{B_n|f|}{t}]^{n+1}\, d\mu,$$

which completes the proof.

Theorem 2. *Let $\Gamma = \{T(t_1, \cdots, t_d);\ t_1, \cdots, t_d > 0\}$ be a d-parameter semigroup of Dunford-Schwartz operators on $L_1(\Omega)+L_\infty(\Omega)$ which is assumed to be strongly continuous with respect to the norm topology of $L_1(\Omega)$. For $f \in L_1(\Omega)+L_\infty(\Omega)$, define*

$$f^*(\omega) = \sup_{\alpha_1, \cdots, \alpha_d > 0} \frac{1}{\alpha_1 \cdots \alpha_d} \Big| \int_0^{\alpha_1} \cdots \int_0^{\alpha_d} T(t_1, \cdots, t_d) f(\omega)\, dt_1 \cdots dt_d \Big|.$$

Then for each $k \ge d-1$ there corresponds a constant $C_k(d) > 0$ so that

(i) *if* $k \geq d$ *then for every* $f \in R_k(\Omega)$ *and all* $t > 0$

$$\int_{\{f^* > t\}} \frac{f^*}{t} [\log \frac{f^*}{t}]^{k-d} \, d\mu \leq \int_{\{C_k(d)|f| > t\}} \frac{C_k(d)|f|}{t} [\log \frac{C_k(d)|f|}{t}]^k \, d\mu \; (<\infty),$$

(ii) *if* $k = d-1$ *then for every* $f \in R_{d-1}(\Omega)$ *and all* $t > 0$

$$\mu\{f^* > t\} \leq \int_{\{C_{d-1}(d)|f| > t\}} \frac{C_{d-1}(d)|f|}{t} [\log \frac{C_{d-1}(d)|f|}{t}]^{d-1} \, d\mu \; (<\infty).$$

Consequently $f \in R_k(\Omega)$ *with* $k \geq d$ *implies* $f^* \in R_{k-d}(\Omega)$.

Proof. We proceed by induction on d.

First suppose that $d=1$. It is then known by [7] that there exists a one-parameter semigroup $\{\tau_1(t_1) \; ; \; t_1 > 0\}$ of positive Dunford-Schwartz operators on $L_1(\Omega) + L_\infty(\Omega)$, strongly continuous with respect to the norm topology of $L_1(\Omega)$, such that for every $f \in L_1(\Omega) + L_\infty(\Omega)$ and all $t_1 > 0$

$$|T(t_1)f| \leq \tau_1(t_1)|f| \quad \text{a. e. on } \Omega.$$

Thus, for $f \in L_1(\Omega) + L_\infty(\Omega)$, if we set

$$M^{\sim} f(\omega) = \sup_{\alpha > 0} \frac{1}{\alpha} \int_0^\alpha \tau_1(t_1)|f|(\omega) \, dt_1,$$

then we have

$$f^* \leq M^{\sim} f \quad \text{a. e. on } \Omega.$$

Since $M^{\sim}$ is a maximal operator on $L_1(\Omega) + L_\infty(\Omega)$ with $A=1$ (cf. [4] or [5]), we observe by Lemma 1 and Theorem 1 that the theorem holds for $d=1$.

Next let us assume that the theorem holds for $d = i-1$. To show that the theorem holds for $d = i$, we define for each $n \geq 1$ an i-parameter semigroup $\Gamma_n = \{T_n(t_1, \cdots, t_i) \; ; \; t_1, \cdots, t_i \geq 0\}$ of Dunford-Schwartz operators on $L_1(\Omega) + L_\infty(\Omega)$ by the relation

$$T_n(t_1, \cdots, t_i) = \begin{cases} I & \text{if } t_1 = t_2 = \cdots = t_i = 0 \\ T(t_1 + u_1/n, \cdots, t_i + u_i/n) & \text{otherwise} \end{cases}$$

where $u_k = (t_1 + \cdots + t_i) - t_k$ for $1 \leq k \leq i$. Put, for $f \in L_1(\Omega) + L_\infty(\Omega)$,

$$M_n f(\omega) = \sup_{\alpha_1, \cdots, \alpha_i > 0} \frac{1}{\alpha_1 \cdots \alpha_i} \left| \int_0^{\alpha_1} \cdots \int_0^{\alpha_i} T_n(t_1, \cdots, t_i) f(\omega) \, dt_1 \cdots dt_i \right|.$$

Then clearly we have

$$f^*(\omega) \leq \liminf_n M_n f(\omega) \quad \text{a. e. on } \Omega,$$

and hence by Fatou 's lemma it is sufficient to observe that the inequalities of the theorem hold, replacing f^* by $M_n f$.

For this purpose we next define, for each $n \geq 1$, an $(i-1)$-parameter semigroup $\Delta_n = \{S_n(t_2, \cdots, t_i) ; t_2, \cdots, t_i > 0\}$ of Dunford-Schwartz operators on $L_1(\Omega) + L_\infty(\Omega)$ by the following relation

$$S_n(t_2, \cdots, t_i) = T_n(0, t_2, \cdots, t_i).$$

Let us denote by $\{\tau_n(t_1) ; t_1 > 0\}$ a one-parameter semigroup of positive Dunford-Schwartz operators on $L_1(\Omega) + L_\infty(\Omega)$, strongly continuous with repect to the norm topology of $L_1(\Omega)$, such that for every $f \in L_1(\Omega) + L_\infty(\Omega)$ and all $t_1 > 0$

$$| T_n(t_1) f | \leq \tau_n(t_1) | f | \quad \text{a. e. on } \Omega,$$

where we let $T_n(t_1) = T_n(t_1, 0, \cdots, 0)$ for $t_1 > 0$. Then for $f \in L_1(\Omega) + L_\infty(\Omega)$ and $\alpha_1, \cdots, \alpha_i > 0$ we have

$$\begin{aligned}
&\frac{1}{\alpha_1 \cdots \alpha_i} \left| \int_0^{\alpha_1} \cdots \int_0^{\alpha_i} T_n(t_1, \cdots, t_i) f \, dt_1 \cdots dt_i \right| \\
&= \frac{1}{\alpha_1} \left| \int_0^{\alpha_1} T_n(t_1) \left[\frac{1}{\alpha_2 \cdots \alpha_i} \int_0^{\alpha_2} \cdots \int_0^{\alpha_i} S_n(t_2, \cdots, t_i) f \, dt_2 \cdots dt_i \right] dt_1 \right| \\
&\leq \frac{1}{\alpha_1} \int_0^{\alpha_1} \tau_n(t_1) \left| \frac{1}{\alpha_2 \cdots \alpha_i} \int_0^{\alpha_2} \cdots \int_0^{\alpha_i} S_n(t_2, \cdots, t_i) f \, dt_2 \cdots dt_i \right| dt_1
\end{aligned}$$

Therefore if $f \in R_k(\Omega)$ and $k \geq i-1$ then the function g_n defined by

$$g_n(\omega) = \sup_{\alpha_2, \cdots, \alpha_i > 0} \frac{1}{\alpha_1 \cdots \alpha_i} \left| \int_0^{\alpha_2} \cdots \int_0^{\alpha_i} S_n(t_2, \cdots, t_i) f(\omega) \, dt_2 \cdots dt_i \right|$$

is, by induction hypothesis, in $R_{k-i+1}(\Omega)$, and for every $t > 0$

$$\begin{aligned}
&\int_{\{g_n > t\}} \frac{g_n}{t} \left[\log \frac{g_n}{t}\right]^{k-i+1} d\mu \\
&\leq \int_{\{C_k(i-1)|f| > t\}} \frac{C_k(i-1)|f|}{t} \left[\log \frac{C_k(i-1)|f|}{t}\right]^k d\mu < \infty,
\end{aligned}$$

and thus if we set

$$M_n^- g_n(\omega) = \sup_{\alpha > 0} \frac{1}{\alpha} \int_0^\alpha \tau_n(t_1) g_n(\omega) \, dt_1,$$

then $M_n f \leq M_n^- g_n$ a. e. on Ω, and furthermore we have:

(i) if $k \geq i$ and $f \in R_k(\omega)$ then for every $t > 0$

$$\int_{\{M_n^- g_n > t\}} \frac{M_n^- g_n}{t} [\log \frac{M_n^- g_n}{t}]^{k-i} d\mu$$
$$\leq \int_{\{C_{k-i+1}(1)g_n > t\}} \frac{C_{k-i+1}(1)g_n}{t} [\log \frac{C_{k-i+1}(1)g_n}{t}]^{k-i+1} d\mu,$$

(ii) if $k=i-1$ and $f \in R_{i-1}(\Omega)$ then for every $t>0$

$$\mu\{M_n^- g_n > t\} \leq \int_{\{C_0(1)g_n > t\}} \frac{C_0(1)g_n}{t} d\mu.$$

Therefore, replacing f^* by $M_n f$, the inequalities of the theorem hold with $C_k(i) = C_k(i-1)C_{k-i+1}(1)$, and so the theorem holds for $d=i$.

The proof is completed.

Remark. It may be readily seen from the above-given argument that if $1<p<\infty$ and $f \in L_p(\Omega)$ then the function f^* of Theorem 2 is in $L_p(\Omega)$ and also satisfies

$$\int |f^*|^p d\mu \leq [\frac{p2^p}{p-1}]^d \int |f|^p d\mu.$$

4. Ergodic theorems.

Theorem 3. *Let $\Gamma = \{T(t_1, \cdots, t_d) ; t_1, \cdots, t_d > 0\}$ be a d-parameter semigroup of Dunford-Schwartz operators on $L_1(\Omega)+L_\infty(\Omega)$ which is assumed to be strongly continuous with respect to the norm topology of $L_1(\Omega)$. If $1 \leq p < \infty$ and $f \in L_p(\Omega)$, then $T(t_1, \cdots, t_d)f$ converges in the norm topology of $L_p(\Omega)$ as $t_1 \longrightarrow 0, \cdots, t_d \longrightarrow 0$ independently.*

Proof. Put, for $t>0$,

$$S(t) = T(t, \cdots, t).$$

Since $\Delta = \{S(t) ; t>0\}$ is a one-parameter semigroup of Dunford-Schwartz operators on $L_1(\Omega)+L_\infty(\Omega)$ which is strongly continuous with repect to the norm topology of $L_1(\Omega)$, it follows from [6] together with an approximation argument that if $1 \leq p < \infty$ and $f \in L_p(\Omega)$ then $S(t)f$ converges in the norm topology of $L_p(\Omega)$ as $t \longrightarrow 0$. Write

$$f_0 = \lim_{t \to 0} S(t)f \quad (\in L_p(\Omega)),$$

then we have $S(t)f_0 = S(t)f$ for all $t>0$, and thus if $t_1, \cdots, t_d > t > 0$ then we have

$$T(t_1, \cdots, t_d)f = T(t_1 - t, \cdots, t_d - t)S(t)f$$
$$= T(t_1 - t, \cdots, t_d - t)S(t)f_0 = T(t_1, \cdots, t_d)f_0.$$

Therefore for each fixed $a > 0$, it follows that

$$\|T(t_1, \cdots, t_d)f - f_0\|_p = \|T(t_1, \cdots, t_d)f_0 - f_0\|_p$$
$$\leq \|T(t_1, \cdots, t_d)S(a)f_0 - S(a)f_0\|_p + \|S(a)f_0 - f_0\|_p$$
$$+ \|T(t_1, \cdots, t_d)[f_0 - S(a)f_0]\|_p.$$

Since $\|S(a)f_0 - f_0\|_p \longrightarrow 0$ as $a \longrightarrow 0$, given an $\varepsilon > 0$ there exists an $a > 0$ so that $\|S(a)f_0 - f_0\|_p < \varepsilon$. Then we get

$$\|T(t_1, \cdots, t_d)f - f_0\|_p < \|T(t_1, \cdots, t_d)S(a)f_0 - S(a)f_0\|_p + 2\varepsilon.$$

On the other hand, since $\Gamma = \{T(t_1, \cdots, t_d)\,;\, t_1, \cdots, t_d > 0\}$ is strongly continuous with respect to the norm topology of $L_p(\Omega)$ for $1 \leq p < \infty$, it follows that

$$\|T(t_1, \cdots, t_d)S(a)f_0 - S(a)f_0\|_p \longrightarrow 0$$

as $t_1 \longrightarrow 0, \cdots, t_d \longrightarrow 0$ independently. Therefore $T(t_1, \cdots, t_d)f$ converges to f_0 in the norm topology of $L_p(\Omega)$ as $t_1 \longrightarrow 0, \cdots, t_d \longrightarrow 0$ independently.

The proof is complete.

Theorem 4. *Let $\Gamma = \{T(t_1, \cdots, t_d)\,;\, t_1, \cdots, t_d > 0\}$ be a d-parameter semigroup of Dunford-Schwartz operators on $L_1(\Omega) + L_\infty(\Omega)$ which is assumed to be strongly continuous with respect to the norm topology of $L_1(\Omega)$. If $f \in R_{d-1}(\Omega)$ then the averages*

$$A(\alpha_1, \cdots, \alpha_d)f(\omega) = \frac{1}{\alpha_1 \cdots \alpha_d} \int_0^{\alpha_1} \cdots \int_0^{\alpha_d} T(t_1, \cdots, t_d)f(\omega)\, dt_1 \cdots dt_d$$

converge almost everywhere on Ω as $\alpha_1 \longrightarrow 0, \cdots, \alpha_d \longrightarrow 0$ independently.

Proof. Theorem 3 ensures us to define an operator T_0 on $L_1(\Omega)$ by the relation

$$T_0 f = \lim_{t_1, \cdots, t_d \to 0} T(t_1, \cdots, t_d)f \qquad (f \in L_1(\Omega))$$

where the limit is in the norm topology of $L_1(\Omega)$ and where $t_1, \cdots, t_d$ tend to zero independently. Then we have $\|T_0\|_1 \leq 1$ and furthermore $\|T_0 f\|_\infty \leq \|f\|_\infty$ for every $f \in L_1(\Omega) \cap L_\infty(\Omega)$. Thus, as in Section 2, we may and will assume that T_0 is a Dunford-Schwartz operator on $L_1(\Omega) \cap L_\infty(\Omega)$.

It will be proved that if $f \in R_{d-1}(\Omega)$ then

$$A(\alpha_1, \cdots, \alpha_d)f(\omega) \longrightarrow T_0 f(\omega) \quad \text{a. e. on } \Omega$$

as $\alpha_1 \longrightarrow 0, \cdots, \alpha_d \longrightarrow 0$ independently.

To do this, first suppose that $1<p<\infty$ and $f\in L_p(\Omega)$. Let, for each $n\geq 1$,

$$f_n=(n^d)\int_0^{1/n}\cdots\int_0^{1/n}T(t_1,\cdots,t_d)f\,dt_1\cdots dt_t\ (\in L_p(\Omega)).$$

Then we see that

$$\lim_n \|f_n-T_0f\|_p=0.$$

Furthermore it may be readily seen that for almost all $(t_1,\cdots,t_d,\omega)\in R_+^d\times\Omega$ with respect to the product of the Lebesgue measure and μ we have

$$T(t_1,\cdots,t_d)f_n(\omega)=(n^d)\int_0^{1/n}\cdots\int_0^{1/n}T(t_1+s_1,\cdots,t_d+s_d)f(\omega)\,ds_1\cdots ds_d$$

where of course $T(t_1,\cdots,t_d)f_n(\omega)$ denotes a scalar representation of $T(t_1,\cdots,t_d)f_n$, $(t_1,\cdots,t_d)\in R_+^d$. Thus for almost all $\omega\in\Omega$, $T(t_1,\cdots,t_d)f_n(\omega)$ as a function of $(t_1,\cdots,t_d)\in R_+^d$ is continuous, and clearly

$$A(\alpha_1,\cdots,\alpha_d)f_n(\omega)\longrightarrow f_n(\omega)\quad \text{a. e. on } \Omega$$

as $\alpha_1\longrightarrow 0,\cdots,\alpha_d\longrightarrow 0$ independently. Since $T_0f_n=f_n$, it then follows that

$$\begin{aligned}&\limsup_{\alpha_1,\cdots,\alpha_d\to 0}|A(\alpha_1,\cdots,\alpha_d)f(\omega)-T_0f(\omega)|\\ &\quad\leq \limsup_{\alpha_1,\cdots,\alpha_d\to 0}|A(\alpha_1,\cdots,\alpha_d)(f-f_n)(\omega)-T_0(f-f_n)(\omega)|\\ &\quad\leq \sup_{\alpha_1,\cdots,\alpha_d>0}|A(\alpha_1,\cdots,\alpha_d)(f-f_n)(\omega)|+|T_0(f-f_n)(\omega)|\\ &\quad\leq (f-f_n)^*(\omega)+|T_0(f-f_n)(\omega)|\quad \text{a. e. on } \Omega.\end{aligned}$$

Since $\lim_n\|(f-f_n)^*\|_p=0$ by the remark in the preceding section and $\lim_n\|T_0(f-f_n)\|_p=\lim_n\|f-f_n\|_p=0$, this implies that for $f\in L_p(\Omega)$ with $1<p<\infty$, $A(\alpha_1,\cdots,\alpha_d)f(\omega)$ converges to $T_0f(\omega)$ a. e. on Ω as $\alpha_1\longrightarrow 0,\cdots,\alpha_d\longrightarrow 0$ independently.

Next suppose that $f\in R_{d-1}(\Omega)$, and then take $f_n\in L_p(\Omega)$, where $1<p<\infty$, so that $|f-f_n|\leq|f|$ and $\lim_n f_n=f$ a. e. on Ω. Then

$$\begin{aligned}&\limsup_{\alpha_1,\cdots,\alpha_d\to 0}|A(\alpha_1,\cdots,\alpha_d)f(\omega)-T_0f(\omega)|\\ &\quad\leq \limsup_{\alpha_1,\cdots,\alpha_d\to 0}|A(\alpha_1,\cdots,\alpha_d)(f-f_n)(\omega)-T_0(f-f_n)(\omega)|\\ &\quad\leq (f-f_n)^*(\omega)+|T_0(f-f_n)(\omega)|\quad \text{a. e. on } \Omega,\end{aligned}$$

and by Theorem 2, for every $t>0$

$$\mu\{(f-f_n)^*>t\} \leq \int_{\{C_{d-1}(d)|f-f_n|>t\}} \frac{C_{d-1}(d)|f-f_n|}{t}\left[\log\frac{C_{d-1}(d)|f-f_n|}{t}\right]^{d-1} d\mu$$

where the right-hand side of the last inequality tends to zero as $n \longrightarrow \infty$, by virtue of Lebesgue's dominated convergence theorem. On the other hand, as in Lemma 1, we have for every $t>0$

$$\mu\{|T_0(f-f_n)|>t\} \leq \frac{2}{t}\int_{\{2|f-f_n|>t\}} |f-f_n|\, d\mu,$$

and the right-hand side of this inequality tends to zero as $n \longrightarrow \infty$, by Lebesgue's convergence theorem, too. Therefore we observe that the theorem holds for $f\in R_{d-1}(\Omega)$, and the proof is completed.

Remark. It is known (cf. [9]) that if $d=1$ then Theorem 4 holds for every $f\in L_1(\Omega)+L_\infty(\Omega)$. But, as is well-known (cf. [4] or [10]), if $d\geq 2$ then the theorem may fail to hold for some $f\in L_1(\Omega)$.

Lemma 2. *Let $\Gamma=\{T(t_1, \cdots, t_d)\,;\, t_1, \cdots, t_d>0\}$ be a d-parameter semigroup of Dunford-Schwartz operators on $L_1(\Omega)+L_\infty(\Omega)$ which is assumed to be strongly continuous with respect to the norm topology of $L_1(\Omega)$. If $1<p<\infty$ and $f\in L_p(\Omega)$, then the averages $A(\alpha_1, \cdots, \alpha_d)f$ converge in the norm topology of $L_p(\Omega)$ as $\alpha_1 \longrightarrow \infty, \cdots, \alpha_d \longrightarrow \infty$ independently.*

Proof. $\{A(\alpha_1, \cdots, \alpha_d)\,;\, \alpha_1, \cdots, \alpha_d>0\}$ may and will be regarded as a net of bounded linear operators on $L_p(\Omega)$. Then it follows that this net is Γ-ergodic in the sense of [8], and since $L_p(\Omega)$ with $1<p<\infty$ is a reflexive Banach space, it follows from [8] that $A(\alpha_1, \cdots, \alpha_d)$ converges in the strong operator topology as $\alpha_1 \longrightarrow\infty, \cdots, \alpha_d \longrightarrow\infty$ independently. This completes the proof.

Theorem 5. *Let $\Gamma=\{T(t_1, \cdots, t_d)\,;\, t_1, \cdots, t_d>0\}$ be a d-parameter semigroup of Dunford-Schwartz operators on $L_1(\Omega)+L_\infty(\Omega)$ which is assumed to be strongly continuous with respect to the norm topology of $L_1(\Omega)$. If $f\in R_{d-1}(\Omega)$ then the averages $A(\alpha_1, \cdots, \alpha_d)f(\omega)$ converge almost everywhere on Ω as $\alpha_1 \longrightarrow \infty, \cdots, \alpha_d \longrightarrow \infty$ independently.*

Proof. Let $1<p<\infty$. Lemma 2 enables us to define an operator T_∞ on $L_p(\Omega)$ by the relation

$$T_\infty f = \lim_{\alpha_1, \cdots, \alpha_d \to\infty} A(\alpha_1, \cdots, \alpha_d)f \qquad (f\in L_p(\Omega))$$

where the limit is in the norm topology of $L_p(\Omega)$ and where $\alpha_1, \cdots, \alpha_d$ tend to infinity independently. Then we have $\|T_\infty\|_p \leq 1$, and for $f \in L_1(\Omega) \cap L_p(\Omega)$ there exists a sequence (f_n) in the set $\{A(\alpha_1, \cdots, \alpha_d)f : \alpha_1, \cdots, \alpha_d > 0\}$ such that

$$T_\infty f = \lim_n f_n \qquad \text{a. e. on } \Omega.$$

Since $\|f_n\|_1 \leq \|f\|_1$ for each $n \geq 1$, it follows from Fatou's lemma that

$$\|T_\infty f\|_1 \leq \liminf_n \|f_n\|_1 \leq \|f\|_1.$$

Hence T_∞ can be uniquely extended to a linear contraction on $L_1(\Omega)$, which satisfies $\|T_\infty f\|_\infty \leq \|f\|_\infty$ for every $f \in L_1(\Omega) \cap L_\infty(\Omega)$. Therefore, as in Section 2, we may and will assume that T_∞ is a Dunford-Schwartz operator on $L_1(\Omega) + L_\infty(\Omega)$.

It will be proved that if $f \in R_{d-1}(\Omega)$ then

$$A(\alpha_1, \cdots, \alpha_d) f(\omega) \longrightarrow T_\infty f(\omega) \qquad \text{a. e. on } \Omega$$

as $\alpha_1 \longrightarrow \infty, \cdots, \alpha_d \longrightarrow \infty$ independently.

To do this, however, in view of the proof of Theorem 4, it is enough to check that the theorem holds for every $f \in L_p(\Omega)$. And to check this it is also enough to notice that the theorem holds for every f in a dense linear manifold of $L_p(\Omega)$.

For this purpose, let M denote the linear manifold generated by the functions f of the form $f = h + [g - T(s_1, \cdots, s_d)g]$, where h, $g \in L_p(\Omega)$, $T(t_1, \cdots, t_d)h = h$ for all $t_1, \cdots, t_d > 0$, and $g \in L_\infty(\Omega)$. Lemma 2 implies that M is dense in $L_p(\Omega)$ with respect to the norm topology of $L_p(\Omega)$, and for such a function f it follows easily that $T_\infty f = h$ and that

$$A(\alpha_1, \cdots, \alpha_d) f(\omega) \longrightarrow h(\omega) \qquad \text{a. e. on } \Omega$$

as $\alpha_1 \longrightarrow \infty, \cdots, \alpha_d \longrightarrow \infty$ independently. This completes the proof.

Let $\Gamma_j = \{T_j(t) ; t > 0\}$, $1 \leq j \leq d$, be one-parameter semigroups of Dunford-Schwartz operators on $L_1(\Omega) + L_\infty(\Omega)$ which are assumed to be strongly continuous with respect to the norm topology of $L_1(\Omega)$. (Here we do not assume that these one-parameter semigroups commute.) Then, since for each $f \in L_p(\Omega)$, with $1 \leq p < \infty$, the function $T_1(t_1) \cdots T_d(t_d) f$ of $(t_1, \cdots, t_d) \in R_+^d$ is continuous with respect to the norm topology of $L_p(\Omega)$, it follows, as in Section 2, that for every $f \in L_1(\Omega) + L_\infty(\Omega)$ there exists a scalar function $T_1(t_1) \cdots T_d(t_d) f(\omega)$, defined on $R_+^d \times \Omega$ and measurable with respect to the product of the Lebesgue measurable subsets of R_+^d and $\mathscr{F}$, such that for each fixed $(t_1, \cdots, t_d) \in R_+^d$, $T_1(t_1) \cdots T_d(t_d) f(\omega)$ as a function of $\omega \in \Omega$ belongs to the equivalence class of $T_1(t_1) \cdots T_d(t_d) f$. Then we may define, for almost all $\omega \in \Omega$,

$$A(\alpha_1, \cdots, \alpha_d)f(\omega)=\frac{1}{\alpha_1\cdots\alpha_d}\int_0^{\alpha_1}\cdots\int_0^{\alpha_d} T_1(t_1)\cdots T_d(t_d)f(\omega)\, dt_1\cdots dt_d$$

Then we have the following theorem which is similar to the above Theorem 5 and a generalization of Theorem 5 in Fava [4].

Theorem 6. *Let $\Gamma_j=\{T_j(t)\,;t>0\}$, $1\leq j\leq d$, be one-parameter semigroups of Dunford-Schwartz operators on $L_1(\Omega)+L_\infty(\Omega)$ which are assumed to be strongly continuous with respect to the norm topology of $L_1(\Omega)$. If $f\in R_{d-1}(\Omega)$ then the averages $A(\alpha_1, \cdots, \alpha_d)f(\omega)$ converge almost everywhere on Ω as $\alpha_1\longrightarrow\infty, \cdots, \alpha_d\longrightarrow\infty$ independently.*

Proof. It is known (cf. [3], p. 694) that if $1<p<\infty$ and $f\in L_p(\Omega)$ then the averages $A(\alpha_1, \cdots, \alpha_d)f(\omega)$ converge almost everywhere on Ω and as well in the norm topology of $L_p(\Omega)$ as $\alpha_1\longrightarrow\infty, \cdots, \alpha_d\longrightarrow\infty$ independently. Thus, by using Theorem 1 repeatedly, we may see, as in Theorem 5, that the desired result holds for $f\in R_{d-1}(\Omega)$. We omit the details.

In conclusion we should like to remark that Yoshimoto [11] has obtained, using a maximal ergodic theorem due to Hasegawa-Sato-Tsurumi [5], vector valued ergodic theorems in the same direction for a one-parameter semigroup $\{T(t)\,;\,t>0\}$ of linear operators on $L_1(\Omega, X)+L_\infty(\Omega, X)$ which satisfies some norm and integrability conditions, X being a reflexive Banach space. Since the scalar field is a reflexive Banach space, Yoshimoto's results generalize ours when restricted to one-parameter semigroups. But we could not extend his results to d-parameter semigroups with $d\geq 2$, because the existence of a *positive* one-parameter semigroup is not known which dominates a given $L_1(\Omega, X)$ contraction operator one-parameter semi group.

Added in proof. Professor S. A. McGrath kindly informed me that he proved, in his recent paper [Local ergodic theorems for noncommuting semigroups, Proc. Amer. Math. Soc. **79** (1980), 212–216], the following local ergodic theorem:

Let $\Gamma_j=\{T_j(t)\,;\,t>0\}$, $1\leq j\leq d$, be as in Theorem 6. Assume, in addition, that $\lim_{t\to 0}\|T_j(t)-I\|_1=0$ for each j. Then for any $f\in R_{d-1}(\Omega)$ the averages $A(\alpha_1, \cdots, \alpha_d)f(\omega)$ converge almost everywhere on Ω as $\alpha_1\longrightarrow 0, \cdots, \alpha_d\longrightarrow 0$ independently.

Modifying his argument and using the local ergodic theorem in [6], it is easily seen that McGrath's theorem holds, without the additional hypothesis that $\lim_{t\to 0}\|T_j(t)-I\|_1=0$ for each j.

REFERENCES

[1] R. V. CHACON and U. KRENGEL : Linear modulus of a linear operator, Proc. Amer. Math. Soc. **15** (1964), 553—559.

[2] N. DUNFORD and J. T. SCHWARTZ : Convergence almost everywhere of operator averages, J. Rational Mech. Anal. **5** (1956), 129—178.

[3] N. DUNFORD and J. T. SCHWARTZ : Linear Operators. Part I, Interscience, Mew York, 1958.

[4] N. A. FAVA : Weak type inequalities for product operators, Studia Math. **42** (1972), 271—288.

[5] S. HASEGAWA, R. SATO and S. TSURUMI : Vector valued ergodic theorems for a one-parameter semigroup of linear operators, Tôhoku Math. J. **30** (1978), 95—106.

[6] R. SATO : On a local ergodic theorem, Studia Math. **58** (1976), 1—5.

[7] R. SATO : Contraction semigroups in Lebesgue space, Pacific J. Math. **78** (1978), 251—259.

[8] R. SATO : On abstract mean ergodic theorems, Tôhoku Math. J. **30** (1978), 575—581.

[9] R. SATO : Two local ergodic theorems on L_∞, J. Math. Soc. Japan **32** (1980), 415—423.

[10] T. R. TERRELL : Local ergodic theorems for N-parameter semigroups of operators, Lecture Notes in Math., No. 160, pp. 262—278, Springer, Berlin, 1970.

[11] T. YOSHIMOTO : Vector valued ergodic theorems for operators satisfying norm conditions, Pacific J. Math. **85** (1979), 485—499.

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