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SOME COMMUTATIVITY THEOREMS FOR n-TORSION FREE RINGS

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Throughout the present note, R will represent an associative ring (with or without 1), and N the set of all nilpotent elements in R. Given $a, b \in R$, we set $[a, b] = ab - ba$, and write $a + ab$ (resp. $a + ba$) formally as $a(1 + b)$ (resp. $(1 + b)a$). If there is a b' such that $b + b' + bb' = b + b' + b'b = 0$, we write $a + b'a + ab + b'ab$ as $(1+b)^{-1}a(1+b)$. Following [3], a ring R is called *s-unital* if for each x in R, $x \in Rx \cap xR$. As stated in [3], if R is an *s-unital* ring, then for any finite subset F of R, there exists an element e in R such that $ex = xe = x$ for all x in F. Such an element e will be called a *pseudo-identity* of F.

Our objective is to prove the following theorems.

Theorem 1. *Let n be a fixed positive integer, and let R be an s-unital ring. Suppose that every commutator $[x, y]$ in R is n-torsion free and $[\{x(1+u)\}^n - x^n(1+u)^n, x] = 0$ for all $u \in N$ and $x \in R$. If, further, R satisfies the polynomial identity $[x^n, y^n] = 0$, then R is commutative.*

Theorem 2. *Let $m \geq n \geq 1$ be fixed integers with $mn > 1$, and let R be an s-unital ring. Suppose that every commutator $[x, y]$ in R is $n!$-torsion free. If, further, R satisfies the polynomial identity $[x^m, y] - [x, y^n] = 0$, then R is commutative.*

In preparation for the proofs of our theorems, we first recall the following lemmas.

Lemma 1 ([1, Lemma 1]). *Let m, n be fixed positive integers.*

(1) *If $[a, [a, b]] = 0$ then $[a^n, b] = na^{n-1}[a, b]$, where $a, b \in R$.*

(2) *Let e be a pseudo-identity of $\{a, b\} \subseteq R$. If $a^m b = 0 = (a+e)^m b$ then $b = 0$.*

(3) *If R satisfies the polynomial identity $[x^n, y^n] = 0$, then the commutator ideal $D(R)$ of R is contained in N.*

Lemma 2 ([1, Lemma 2]). *Let m, n be fixed positive integers, and let R be an s-unital ring in which every commutator is n-torsion free.*

(1) *If* $nx^m[x, a] = 0$ *for all* $x \in R$, *then* $[x, a] = 0$.

(2) *If* R *satisfies the polynomial identity* $[x^n, y] = 0$, *then* R *is commutative.*

Lemma 3. *Let* n *be a fixed positive integer, and let* R *be an s-unital ring in which every commutator is n-torsion free. If* R *satisfies the polynomial identity* $[x^n, y^n] = 0$, *then* $[u, x^n] = 0$ *and* $[u, v] = 0$ *for all* $u, v \in N$ *and* $x \in R$.

Proof. The first assertion is proved in the proof of [1, Theorem 1]. Then, repeating the same argument, we can prove also the latter.

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Let $u \in N$ and $x \in R$. Then, by Lemma 3, we obtain $[1 + u, \{(1 + u)x\}^n] = 0$. Hence, by hypothesis,

$$0 = x\{(1+u)x\}^n - x(1+u)^{-1}\{(1+u)x\}^n(1+u) = \{x(1+u)\}^n x - x\{x(1+u)\}^n$$
$$= [\{x(1+u)\}^n, x] = [x^n(1+u)^n, x] = x^n[(1+u)^n, x].$$

Then, since every pseudo-identity of $\{x, u\}$ is that of $\{x, [(1+u)^n, x]\}$, Lemma 1 (2) shows that $[(1+u)^n, x] = 0$ for all $x \in R$. Moreover, by Lemma 1 (3), $[1+u, x] = [u, x] \in N$, and hence by Lemma 3 we see that $[1+u, [1+u, x]] = 0$. Now, by Lemma 1 (1), $n(1+u)^{n-1}[u, x] = [(1+u)^n, x] = 0$, whence it follows $[u, x] = 0$. We have thus shown that N is contained in the center Z of R.

To complete the proof, let $x, y \in R$. Since $[x, y] \in N \subseteq Z$ by Lemma 1 (3) and the above, there holds $nx^{n-1}[x, y^n] = 0$ (Lemma 1 (1)). Hence, by Lemma 2 (1), $[x, y^n] = 0$. Now, R is commutative by Lemma 2 (2).

It was shown in [1] that in an s-unital ring in which every commutator is $n(n-1)$-torsion free ($n > 1$), the identity $(xy)^n = x^n y^n$ implies the identity $[x^n, y^n] = 0$. In view of this, we obtain Theorem 2 in [1] as a corollary to Theorem 1.

Finally, we shall prove Theorem 2.

Proof of Theorem 2. If $n = 1$, then $m > 1$ and, by hypothesis, we see that R satisfies the identity $[x - x^m, y] = 0$. Hence, by a well known theorem of Herstein [2], R is commutative. So, henceforth we may assume $n > 1$. Let $x, y \in R$. By hypothesis, $[x^m, y] = [x, y^n]$. Replacing y by ky, where k is an arbitrary positive integer, we get

$k^n[x, y^n] = k[x^m, y]$, and hence

$$(*) \qquad (k^n - k)[x, y^n] = 0.$$

We show $[x, y^n] = 0$. Suppose not. Then the additive order of $[x, y^n]$ is obviously a positive integer $q\,(> 1)$. Since $[x, y^n]$ is $n!$-torsions free by hypothesis, we see that $(q, n!) = 1$. Let $p\,(> n)$ be a prime factor of q, and $q = pd$. Since $p(p^{n-1} - 1)\,[x, y^n] = 0$ by (*), $q = pd$ divides $p\,(p^{n-1} - 1)$, and so $(p, d) = 1$. As is well known, every ring is a subdirect sum of subdirectly irreducible rings. There exists therefore a homomorphism f of R onto a subdirectly irreducible ring R' such that the order of $f([x, y^n])$ is pd' with a divisor d' of d. If $d' > 1$, then $I_1' = \{r' \in R' \mid pr' = 0\}$ and $I_2' = \{r' \in R' \mid d'r' = 0\}$ are non-zero ideals of the subdirectly irreducible ring R', and hence $I_1' \cap I_2' \neq 0$. But, on the other hand, $(p, d') = 1$ implies $I_1' \cap I_2' = 0$. This contradiction shows that $d' = 1$. Hence, by (*), $k^n - k \equiv 0 \pmod{p}$ for $k = 0, 1, \cdots, n\,(< p)$. But this is impossible. This contradiction proves that $[x, y^n] = 0$ for all $x, y \in R$. Hence, R is commutative again by Lemma 2 (2).

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