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ON INNER (σ, τ)-DERIVATIONS OF SIMPLE RINGS

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Throughout the present note, except the final remark, A will represent a (two-sided) simple ring with 1, C the center of A, and B a (right) primitive proper subring of A containing 1. We consider two ring monomorphisms σ, τ of A into itself, and set $J=\{x\in A \mid \sigma(x)=\tau(x)\}$. Further, we use the following convention: a, a' be arbitrary elements of A, and b, b' elements of B.

Given $x\in A$, the map $\delta_x\colon A\to A$ defined by $\delta_x(a)=x\sigma(a)-\tau(a)x$ is called the *inner* (σ, τ)*-derivation* effected by x. In fact, δ_x is a (σ,τ)-derivation of A: $\delta_x(aa')=\delta_x(a)\sigma(a')+\tau(a)\delta_x(a')$. Recently, in [1], Y. Felix generalized a result of P. Van Praag [6] as follows: *If A is a division ring with $[A:C]>4$ and if B is a proper subdivision ring of A which is invariant relative to all the inner (σ, τ)-derivations effected by elements of A, then B is contained in C.*

In what follows, by making use of the method employed in [3], we shall prove the following theorem that generalizes [3, Theorem 1] and recovers the result of Felix mentioned above.

Theorem. *Let A be a simple ring with* 1, *C the center of A, and B a primitive proper subring of A containing* 1. *Let σ, τ be ring monomorphisms of A into itself, and $J=\{x\in A\mid\sigma(x)=\tau(x)\}$. If B is invariant relative to all the inner (σ, τ)-derivations effected by elements of A, then either B is contained in $C\cap J$, or $[A:C]=4$ and B equals its centralizer $V_A(B)$ in A.*

In preparation for the proof of our theorem, we establish the following lemma.

Lemma. *Assume that B is invariant relative to all the inner (σ, τ)-derivations effected by elements of A.*

(1) $\sigma(b)-\tau(b)\in B$.

(2) $[B, \sigma(B)]=0$ *or* $[B, \tau(B)]=0$.

(3) *B is a field.*

Proof. (1) Obviously, $\sigma(b)-\tau(b)=\delta_1(b)\in B$.

(2) As is easily seen,

$$a[b, \sigma(b')] = \delta_{ab}(b') - \delta_a(b')\, b \in B,$$
$$[b, \tau(b')]\, a = \delta_{ba}(b') - b\delta_a(b') \in B.$$

Hence, $[B, \sigma(B)] \subseteq A[B, \sigma(B)] \subseteq B$ and $[B, \tau(B)] \subseteq [B, \tau(B)]\, A \subseteq B$. Since $A[B, \sigma(B)]\, B[B, \tau(B)]\, A$ is an ideal of the simple ring A and is contained in B, it follows that $[B, \sigma(B)]\, B[B, \tau(B)] = 0$. Now, noting that B is a prime ring, we readily obtain that either $[B, \sigma(B)] = 0$ or $[B, \tau(B)] = 0$.

(3) According to (2), $[B, \sigma(B)] = 0$ or $[B, \tau(B)] = 0$. We consider the case $[B, \sigma(B)] = 0$. Then, by (1), $[\sigma(b) - \tau(b), \sigma(b')] = 0$, namely, $[\sigma(b'), \sigma(b)] = [\sigma(b'), \tau(b)]$. Hence, we see that

$$\begin{aligned}\sigma([b', [b', b]]) &= \sigma(b')[\sigma(b'), \sigma(b)] - [\sigma(b'), \sigma(b)]\sigma(b') \\ &= \sigma(b')[\sigma(b'), \sigma(b)] - [\sigma(b'), \tau(b)]\sigma(b') \\ &= [\sigma(b'), \delta_{\sigma(b')}(b)] = 0,\end{aligned}$$

whence it follows $[b', [b', b]] = 0$. Now, by Kaplansky-Amitsur Theorem (see, e. g. [2, p. 17]), B is a field. Similarly, we can prove the same for the case $[B, \tau(B)] = 0$.

We are now in a position to prove our theorem.

Proof of Theorem. At any rate, B is a field by Lemma (3). We distinguish here two cases: Case 1, B is contained in J; Case 2, B is not contained in J.

Case 1. By Lemma (2), $[B, \sigma(B)] = [B, \tau(B)] = 0$. Since

$$\sigma([[a, b], b]) = [\delta_{\sigma(a)}(b), \sigma(b)] = 0,$$

we obtain $[[a, b], b] = 0$, and therefore

$$[a, b^2] = 2[a, b]\, b,$$
$$2[a, b]\,[a', b] = [[aa', b], b] = 0.$$

Hence, if A is not of characteristic 2, then $[a, b]\,[a', b] = 0$ and

$$[a, b]a'[a, b] = [a, b]a'[a, b] + [a, b]\,[a', b]a = [a, b]\,[a'a, b] = 0,$$

so that $[a, b]\, A\, [a, b] = 0$. Since A is prime, this implies $[a, b] = 0$, and therefore $B \subseteq C$. Henceforth, we assume that A is of characteristic 2 and B is not contained in C. Then,

$$\begin{aligned}\sigma([a', [a, b]^2]) &= [\sigma(a'), (\delta_{\sigma(a)}(b))^2] \\ &= 2[\sigma(a'), \delta_{\sigma(a)}(b)]\, \delta_{\sigma(a)}(b) = 0,\end{aligned}$$

whence it follows $[a', [a, b]^2] = 0$, namely $[a, b]^2 \in C$. Hence, $[A : C] = 4$ by [4, Theorem 2]. Now, choose $a_0 \in A$ and $b_0 \in B$ with $[a_0, b_0] \neq 0$.

Since $[\sigma(a_0), \sigma(b_0)] = \delta_{\sigma(a_0)}(b_0)$ is a non-zero element of B and $[B, \sigma(B)] = 0$, for any $c \in C$ there holds that

$$\begin{aligned} c &= c[\sigma(a_0),\ \sigma(b_0)]^{-1}\ [\sigma(a_0),\ \sigma(b_0)] \\ &= [c[\sigma(a_0),\ \sigma(b_0)]^{-1}\sigma(a_0),\ \sigma(b_0)] \in B, \end{aligned}$$

namely, $C \subsetneq B$. Consequently, we obtain $B = V_A(B)$.

Case 2. Choose an element b^* of B not contained in J. Then $b_0 := \sigma(b^*) - \tau(b^*) = \delta_1(b^*)$ is a non-zero element of B. If v is in $V_A(\sigma(b^*))$, then $b_0 v = \delta_v(b^*) \in B$, so that $v \in B$. In particular, $\sigma(B) \subseteq B$ and $\tau(B) \subseteq B$ (Lemma (1)). Hence, $B \subseteq V_A(B) \subseteq V_A(\sigma(b^*)) \subseteq B$, and then $B = V_A(B) = V_A(\sigma(b^*))$. As is easily seen, $b_0 a = [\sigma(b^*), a] + \delta_a(b^*)$ and $[A, B]$ is a B-B-submodule of A. Thus, we see that $A = [A, B] + B$. This enables us to see that $[A, B]^2 \neq 0$. In fact, $[A, B]^2 = 0$ implies a contradiction that $[A, B]$ is a non-zero nilpotent ideal of A. Let u, u' be elements of $[A, B]$ with $uu' \neq 0$. Since

$$\begin{aligned} [b', a]\,\sigma(b) - \tau(b)\,[b', a] &= [b', a\sigma(b)] - [b', \tau(b)a] \\ &= [b', \delta_a(b)] = 0, \end{aligned}$$

we see that $x\sigma(b) = \tau(b)x$ for all $x \in [A, B]$. Then

$$\begin{aligned} uu' &= u(\sigma(b^*) - \tau(b^*))\, b_0^{-1}u' \\ &= \tau(b^*)ub_0^{-1}u' - ub_0^{-1}u'\,\sigma(b^*) = -\,\delta_{ub_0^{-1}u'}(b^*) \in B. \end{aligned}$$

Hence, uu' is a unit in B, and also $u'u$ is in B and non-zero by $(uu')^2 \neq 0$. We see therefore that u is a unit in A. Now, for any $x \in [A, B]$,

$$u^{-1}x\sigma(b^*) = u^{-1}\tau(b^*)x = u^{-1}\tau(b^*)uu^{-1}x = \sigma(b^*)\,u^{-1}x.$$

This means that $u^{-1}x \in V_A(\sigma(b^*)) = B$, whence it follows $[A, B] = uB$. We have thus seen that $[A: B]_R = 2$. Combining this with $B = V_A(B)$, we readily obtain $[A: C] = 4$ (see, e. g. [5, Proposition 7. 1]).

Remark. (1) Let $A(\ni 1)$ be a primitive ring with non-zero socle, and B a completely primitive proper subring of A containing 1. If B is invariant relative to all the inner (σ, τ)-derivations effected by elements of A then, by slightly modifying the argument used in the proof of [3, Lemma 3 (b)], we can show that $[B, \sigma(B)] = 0$ or $[B, \tau(B)] = 0$. A careful examination of the proofs of Lemma (3) and Theorem shows that our theorem is still valid under the hypothesis of [3, Theorem 2].

(2) Assume that A is a central simple algebra of rank 4 over C

and B is a subfield of A with $B=V_A(B)$. According to [3, Lemma 4], there exists then an inner automorphism ρ of A such that $\rho(b)\in B$ and $ba-a\rho(b)\in B$. If B is invariant relative to all the inner (σ, τ)-derivations effected by elements of A then, by Lemma (1) and (2), we see that $\sigma(B), \tau(B)\subseteq B$. Since $a(\sigma(b)-\rho\tau(b))=\delta_a(b)+(\tau(b)a-a\cdot\rho\tau(b))\in B$ and $B\subsetneq A$, we readily obtain $\sigma(b)-\rho\tau(b)=0$, namely $\sigma=\rho\tau$ on B. Conversely, if $\tau(B)\subseteq B$ and $\sigma=\rho\tau$ on B, then $\delta_a(b)=a\cdot\rho\tau(b)-\tau(b)a\in B$.

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