

NOTE ON AZUMAYA ALGEBRAS AND H-SEPARABLE EXTENSIONS

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Let A/B be a ring extension with common identity 1, and C be the center of A . If $A \otimes_B A$ is A - A -isomorphic to an A - A -direct summand of a finite direct sum A^n then A/B is called to be H -separable. As is well known, A/C is H -separable if and only if A is an Azumaya C -algebra. The purpose of this note is to prove the following theorem, which has an application (Th. 2).

Theorem 1. *Let A be an Azumaya C -algebra, and $A \supset B \supset C$. If A_B is projective then A/B is H -separable.*

Proof. Since A/C is separable, there exists an element $\sum_i r_i \otimes s_i$ in $A \otimes_C A$ such that $\sum_i r_i s_i = 1$ and $\sum_i ar_i \otimes s_i = \sum_i r_i \otimes s_i a$ for all $a \in A$. Further, since A_B is f. g. projective, there exists a finite number of elements $t_j \in A$ and $f_j \in \text{Hom}(A_B, B_B)$ such that $\sum_j t_j f_j(a) = a$ for all $a \in A$. Then, the mapping $\theta: u \otimes v \rightarrow \sum_j ut_j \otimes f_j(v)$ of $A \otimes_C A$ into itself is an endomorphism, and

$$\begin{aligned} \sum_{i,j} r_i t_j \otimes f_j(s_i a x) y &= \theta(\sum_i r_i \otimes s_i a x) y = \theta(\sum_i ar_i \otimes s_i x) y \\ &= \sum_{i,j} ar_i t_j \otimes f_j(s_i x) y \end{aligned}$$

where $a, x, y \in A$. This implies that the map $\phi: A \otimes_B A \rightarrow A \otimes_C A$ defined by $x \otimes y \rightarrow \sum_{i,j} r_i t_j \otimes f_j(s_i x) y$ is an A - A -homomorphism. Obviously, the canonical map $\psi: A \otimes_C A \rightarrow A \otimes_B A$ is an A - A -homomorphism and $\psi\phi$ is the identity map of $A \otimes_B A$. Hence ${}_A A \otimes_B A \llcorner \bigoplus_A A \otimes_C A$. Since A/C is H -separable, it follows that A/B is H -separable.

Next, we need the following

Lemma. *Let A/B be H -separable, and ${}_A M$ a unital A -module. If ${}_B M$ is a generator then so is ${}_A M$.*

Proof. Since ${}_B M$ is a generator, ${}_B B \llcorner \bigoplus_B M^n$ for some integer $n > 0$. Further, since A/B is H -separable, ${}_A A \otimes_B A \llcorner \bigoplus_A A^n$ for some integer $m > 0$. Then, we obtain ${}_A A \simeq {}_A A \otimes_B B \llcorner \bigoplus_A A \otimes_B M^n \simeq {}_A (A \otimes_B M)^n \llcorner \bigoplus_A M^{mn}$.

Now, let B be a commutative ring, G a finite group of automorphisms

of B , and $R = B^G$ (the fixed ring of G in B). Moreover, $\Delta(B; G)$ denotes the trivial crossed product $\bigoplus_{\sigma \in G} Bu_\sigma$ with $u_\sigma u_\tau = u_{\sigma\tau}$ and $u_\sigma b = \sigma(b)u_\sigma$ ($\sigma, \tau \in G, b \in B$). Obviously, the map $j: \Delta(B; G) \rightarrow \text{Hom}(B_R, B_R)$ defined by $j(bu_\sigma)(x) = b\sigma(x)$ ($b, x \in B, \sigma \in G$) is a ring homomorphism. If j is an isomorphism and B_R is f. g. projective then B/R is called to be G -Galois (cf. [1], [2]). Under this situation, we shall prove the following theorem which contain some characterizations of Galois extensions of commutative rings.

Theorem 2. *Let B be a commutative ring, G a finite group of automorphisms of B , $R = B^G$, and $\Delta = \Delta(B; G)$. Then the following conditions are equivalent.*

- (1) B/R is G -Galois.
- (2) Δ is an Azumaya R -algebra.
- (3) Δ/B is H -separable.

When this is the case, B is a maximal commutative R -subalgebra of Δ with $\Delta \otimes_R B \simeq M_m(B)$ and $B \otimes_R \Delta \simeq M_m(B)$, where m is the order of G .

Proof. (1) \implies (2). It is well known ([2, Prop. 3.1.2 and Prop. 2.4.1]). (2) \implies (3). Since ${}_B\Delta$ is free, it follows from Th. 1. (3) \implies (1). By Lemma, ${}_B\Delta$ is a generator. Hence B/R is G -Galois by [1, Prop. A.1]. Finally, if B/R is G -Galois then B coincides with the centralizer of B in Δ , and hence the last assertion follows immediately from [3, Lemma 1 (3)].

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