Math. J. Okayama Univ. 23 (1981), 13—16

ON RINGS WHOSE NON-CONSTANT SEMIGROUP ENDOMORPHISMS ARE RING ENDOMORPHISMS

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Throughout the present note, R will represent a ring (different from 0), J the Jacobson radical of R, and R' the subset $\{xy \mid x, y \in R\}$. If $R = R'$ and $2x = 0 = x^2$ for all $x \in R$, then R is called a *power ring.* Following [3], R is called a *right perfect ring* if J is right T-nilpotent and R/J is Artinian. A ring R is called a *right duo ring* if every right ideal of R is two-sided. As is easily seen, every prime ideal of a right duo ring is completely prime.

In [2], J. Cresp and R. P. Sullivan considered the following property of rings : (ε') every non-constant (multiplicative) semigroup endomorphism is a ring endomorphism, and dealt with commutative rings with the property.

The purpose of this note is to prove the following theorems.

Theorem 1. *If a right perfect ring R has the property (ε'), then R is either* GF(2) *or a zero-ring of order* 2, *and conversely.*

Theorem 2. *Suppose R has the property (ε'). If R is a right duo ring or a P. I.-ring, then there holds one of the following*:

1) *R is a completely prime ring with no non-zero proper prime ideals.*
2) *R is a zero-ring of order* 2.
3) *R is a non-nilpotent, nil ring with $R = R'$.*

In preparation for the proof of our theorems, we establish the following lemmas.

Lemma 1. *Suppose R has the property (ε'). If P is a proper completely prime ideal of R, then $P = 0$.*

Proof. Obviously, the map $f: R \to R$ defined by

$$xf = \begin{cases} 0 & \text{if } x \in P \\ x & \text{if } x \notin P \end{cases}$$

is a non-constant semigroup endomorphism, and so by (ε'), is a ring endomorphism. Hence, $A = (R \setminus P) \cup \{0\}$ is a subring of R. Since $R = P \cup A$ and $R \neq P$, by Brauer's trick we obtain $R = A$, and hence $P = 0$.

Lemma 2 ([2, Theorem 1]). *Let R be a Dedekind finite ring with* 1. *If R has the property* (ε'), *then* $R=\mathrm{GF}(2)$.

Proof. Let N be the set of all non-units in R. As is easily seen, the map $f: R \to R$ defined by

$$xf=\begin{cases} 0 & \text{if } x\in N \\ 1 & \text{if } x\notin N \end{cases}$$

is a non-constant semigroup endomorphism, and so by (ε'), is a ring endomorphism. Hence, N is a proper completely prime ideal, and so $N=0$ by Lemma 1. We conclude therefore that f is an isomorphism of R onto GF(2).

Lemma 3. *If R has the property* (ε'), *then there holds one of the following*:

1) *R is a completely prime ring with no non-zero proper completely prime ideals.*

2) *R is a zero-ring of order* 2.

3) *R is a non-reduced ring with $R=R'$.*

Proof. If R is a reduced ring, then by [1, Theorem 2] R is a subdirect sum of completely prime rings. Hence, by Lemma 1, R is a completely prime ring without non-zero proper completely prime ideals. Next, assume that R is a non-reduced ring with $R\neq R'$. Then there exists a non-zero element a with $a^2=0$. Given a proper subset S of R containing R', we see that the map $f: R\to R$ defined by

$$xf=\begin{cases} 0 & \text{if } x\in S \\ a & \text{if } x\notin S \end{cases}$$

is a non-constant semigroup endomorphism, and so as usual, is a ring endomorphism. Thus, S is an ideal of R as the kernel of f, and R/S is of order 2. Let b be an arbitrary element in $R\backslash R'$. Since both R' and $R'\cup\{b\}$ are ideals of R, we see that $x'+b=b$ for all $x'\in R'$, whence it follows $R'=0$. We conclude therefore that R is a zero-ring of order 2.

We are now ready to complete the proof of our theorems.

Proof of Theorem 1. According to Lemma 3, we consider first the case that R is a completely prime ring. Obviously, R is then a division ring, and therefore $R=\mathrm{GF}(2)$ by Lemma 2. Next, we consider the case that R is a non-reduced ring with $R=R'$. Since $RJ\neq R$ by [3, Lemma 2], J cannot equal R. Let p be the natural homomorphism of R onto R/J, $\overline{M}$ the set of all non-units in R/J, and $M=p^{-1}(\overline{M})$. Let e be an idempotent lifted from the identity of R/J, and define the map $f: R \rightarrow R$ by

$$xf = \begin{cases} 0 & \text{if } x \in M \\ e & \text{if } x \notin M. \end{cases}$$

Then f is a non-constant semigroup endomorphism, and therefore a ring endomorphism. Hence, M is a proper completely prime ideal, and so $M=0$ (and $R=\mathrm{GF}(2)$) by Lemma 1. But this is impossible.

Proof of Theorem 2. We consider first the case that R is a right duo ring. According to Lemma 3, we consider the case that R is a non-reduced ring with $R=R'$. Obviously, R cannot be nilpotent. Moreover, R cannot be completely prime, and so R contains no proper (completely) prime ideals (Lemma 1). Hence, R is a nil ring.

In what follows, we assume that R is a *P. I.*-ring. If R has a proper prime ideal P, then R/P is a Goldie ring (see, e. g. [4, Corollary 1]). Let X be the subset of R consisting of all $x \in R$ such that $x+P$ is regular in R/P. Then we can define a non-constant semigroup endomorphism f by

$$xf = \begin{cases} 0 & \text{if } x \notin X \\ x & \text{if } x \in X. \end{cases}$$

By (ε'), f is a ring endomorphism, and so $\mathrm{Ker}\, f (\supseteq P)$ is a completely prime ideal of R. Then, by Lemma 1, $\mathrm{Ker}\, f=0$ (and so $P=0$) and R is completely prime. Now, our assertion is immediate by Lemma 3.

Finally, by making use of Lemma 2 and Theorem 2, we reprove [5, Theorem 1].

Corollary 1. *If a commutative ring R has the property* (ε'), *then there holds one of the following*:

1) $R=\mathrm{GF}(2)$.
2) *R is a zero-ring of order* 2.
3) *R is a power ring.*

Proof. First, we claim that for any $x, y \in R$, $2xy=0$, $x^2y=0=xy^2$, and x^4 is an idempotent. If $x^2=0$ for all x, then $0=(x+y)^2=2xy$ and $x^2y=xy^2$. Next, assume that $a^2 \neq 0$ for some a. Then $(x+y)^2=x^2+y^2$ by (ε'). Hence, $2xy=0$. If $x^3=0$ for all x, then $0=(x+y)^3=x^2y-xy^2$. On the other hand, if $b^3 \neq 0$ for some b, then $(x+y)^3=x^3+y^3$ by (ε'), whence it follows $0=x^2y-xy^2$. We have therefore proved the claim except the last one. When $y=x^2$, $x^2y=xy^2$ implies $x^4=x^5$. Hence, x^4 is an idempotent.

Now, we proceed to complete the proof. If R is completely prime, then R contains 1 by the above, and hence $R=\mathrm{GF}(2)$ by Lemma 2. Next, assume that R is a non-nilpotent, nil ring with $R=R'$ (see Theorem 2). Then, for every $a=uv\in R'=R$ we have $2a=0$ and $a^2=u^2v^2=u^4v=0$ by the above claim. This completes the proof.

PEFERENCES

[1] V. A. ANDRUNAKIEVIČ and JU. M. RJABUHIN: Rings without nilpotent elements and completely simple ideals, Dokl. Akad. Nauk SSSR **180** (1968), 9—11: Soviet Math. Dokl. **9** (1968), 565—568.

[2] J. CRESP and R. P. SULLIVAN: Semigroups in rings, J. Austral. Math. Soc. **20A** (1975), 172—177.

[3] H. KOMATSU and H. TOMINAGA: On the existence of identities in ideals and subrings, to appear in Math. J. Okayama Univ.

[4] L. ROWEN: Some results on the center of a ring with polynomial identity, Bull. Amer. Math. Soc. **79** (1973), 219—223.

[5] R. P. SULLIVAN: Semigroup endomorphisms of rings, J. Austral. Math. Soc. **26A** (1978), 319—322.

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(Received October 9, 1980)