Math. J. Okayama Univ. **23** (1981), 7—11

# SOME COMMUTATIVITY THEOREMS FOR SEMI-PRIME RINGS. II

To Professor Kentaro Murata on his sixtieth birthday

YASUYUKI HIRANO, MOTOSHI HONGAN
and HISAO TOMINAGA

Throughout the present paper, $R$ ($\neq 0$) will represent an associative ring (with or without 1), and $C$ the center of $R$. Let $N$ be the set of all nilpotent elements of $R$. We denote by $J$ and $D$ the Jacobson radical and the commutator ideal of $R$, respectively. A ring $R$ is called $s$-unital if for each $x$ in $R$, $x \in Rx \cap xR$. As stated in [9], if $R$ is an $s$-unital ring, then for any finite subset $F$ of $R$, there exists an element $e$ in $R$ such that $ex = xe = x$ for all $x$ in $F$. Such an element $e$ will be called a pseudo-identity of $F$.

The present objective is to add to the study in the previous paper [8].

**1.** In this section, we shall prove a commutativity theorem for prime rings which deduces the principal theorem of [1].

**Lemma 1.** *Let $R$ be a prime ring without non-zero nil ideals in which for every pair of elements $x$, $y$ there exists a positive integer $n = n(x, y)$ such that $[(xy)^n - (yx)^n, x] = 0$. If $C$ contains a non-zero element $c$, then for each quasi-regular $a \in R$ and each $x \in R$ there exist positive integers $k$, $l$ such that $[a, [a, x^k]] = 0$ and $l\,!\,[a, x^k]^l = 0$.*

*Proof.* Using hypothesis for elements $c(1+a)$ and $x(1+a)^{-1}$, there exists a positive integer $k$ such that $c^k[(1+a)x^k(1+a)^{-1} - x^k, c(1+a)] = 0$, where $1+a$ is formally invertible. Then

$$c^{k-1}[a, [a, x^k]] = -c^k[(1+a)x^k(1+a)^{-1} - x^k, c(1+a)]\,(1+a) = 0.$$

Since $R$ is prime, we obtain $[a, [a, x^k]] = 0$. Similarly, $[a, [a, (x^{2k})^m]] = 0$ with some positive integer $m$. Hence, by [8, Lemma (3)], $0 = {}_{2m}[a, (x^k)^{2m}] = (2m)\,!\,[a, x^k]^{2m}$.

**Lemma 2.** *Let $R$ be a torsion-free prime ring without non-zero nil ideals in which for each pair of elements $x$, $y$ there exists a positive integer $n = n(x, y)$ such that $[(xy)^n - (yx)^n, x] = 0$. If $C$ contains a non-zero* *element $c$, then $R$ is a reduced ring.*

*Proof.* Let $a$ be an element of $R$ with $a^2 = 0$. Given $x \in R$, there exists a positive integer $n$ such that $[(ax)^n - (xa)^n, a] = 0$. Then we have $2(ax)^{n+1} = [(ax)^n - (xa)^n, a]x = 0$, and therefore $(ax)^{n+1} = 0$. Hence $aR$ is a nil right ideal. Let $u, v \in aR$. Then, by Lemma 1, there exists a positive integer $k$ such that $[u, [u, (v+c)^k]] = 0$. If $v^2 = 0$ then $kc^{k-1}[u, [u, v]] = [u, [u, (v+c)^k]] = 0$, and so $[u, [u, v]] = 0$. By making use of induction on the nilpotency indices of nilpotent elements $v$, we can easily see that $[u, [u, v]] = 0$ for all $u, v \in aR$. Thus, [6, Lemma 2. 1. 1] shows that $a = 0$.

**Theorem 1.** *Let $R$ be a prime ring without non-zero nil ideals in which for each pair of elements $x, y$ there exists a positive integer $n = n(x, y)$ such that $[(xy)^n - (yx)^n, x] = 0$. If $C$ contains a non-zero element, then $R$ is commutative.*

*Proof.* First, we show that $J$ is commutative. Let $a, b \in J$. By Lemma 1, there exist then positive integers $k$, $l$ such that $[a, [a, b^k]] = 0$ and $l\,!\,[a, b^k]^l = 0$. If $R$ is of characteristic $p > 0$, then $[a^p, b^k] = pa^{p-1}[a, b^k] = 0$, and hence $J$ is commutative by [7, Theorem]. On the other hand, if $R$ is of characteristic 0, then $[a, b^k] = 0$ by Lemma 2, and hence $J$ is commutative again by [7, Theorem]. Since $D \subseteq J$ by [8, Corollary 1], $D$ is a commutative ideal. Hence, $D \subseteq C$ by [5, Lemma 1. 5]. Thus, again by the same lemma, we obtain eventually $R = C$.

**Corollary 1** ([1, Theorem 2]). *Let $R$ be a ring without non-zero nil ideals. If for each $x, y \in R$ there exists a positive integer $n = n(x, y)$ such that $(xy)^n - (yx)^n \in C$, then $R$ is commutative.*

*Proof.* As is well known, $R$ is a subdirect sum of prime rings without non-zero nil ideals. Thus, we may assume that $R$ itself is prime. Then, $C$ is non-zero by [3, Theorem 3], and therefore is commutative by Theorem 1.

**2.** In this section, we shall prove a commutativity theorem for $s$-unital semi-prime rings which improves [4, Theorem 1].

**Lemma 3.** *Suppose that for each $x, y \in R$ there exist positive integers $m = m(x, y)$ and $n = n(x, y)$ such that $[(xy)^n, [(xy)^n, (yx)^m]] = 0$. If $a^2 = 0$ then $aR$ is a nil right ideal.*

*Proof.* See the proof of [8, Theorem 1].

**Lemma 4.** *If $R$ is a semi-prime ring satisfying the polynomial identity $[x+y+xy, x+y+yx] = 0$ (or $[xy, yx] = 0$), then $R$ is commutative.*

*Proof.* Since $D$ is a nil ideal by [2, Theorem 1], $D$ must be $0$ by [6, Lemma 2. 1. 1].

**Theorem 2.** *Let $l$, $m$ be fixed positive integers. If $R$ is an s-unital semi-prime ring, then the following are equivalent:*

1) *$R$ is commutative.*

2) *For each $x, y \in R$ there exists a positive integer $n = n(x, y)$ such that $[x^k y^k - (xy)^k, x] = 0$ and $[x^k y^k - (xy)^k, y] = 0$ $(k = n, n+1, n+2)$.*

3) *For each $x, y \in R$ there exists a positive integer $n = n(x, y)$ such that $[y^k x^k - (xy)^k, x] = 0$ and $[y^k x^k - (xy)^k, y] = 0$ $(k = n, n+1, n+2)$.*

4) *For each $x, y \in R$ there exists a positive integer $n = n(x, y)$ such that $[(xy)^k, (yx)^m] = 0$ $(k = n, n+l)$.*

*Proof.* 1) $\Longrightarrow$ 2)—4). Trivial.

2) $\Longrightarrow$ 1). We first prove the case that $R$ is semi-primitive. Note that 2) is inherited by all subrings and homomorphic images of $R$. Note also that no complete matrix ring $(S)_t$ over a division ring $S$ $(t > 1)$ satisfies 2), as a consideration of $x = E_{12}$ and $y = E_{21}$ shows. Because of these facts and the structure theory of primitive rings, we may assume that $R$ is a division ring. Let $x$, $y$ be non-zero elements of $R$. By 2), there exists a positive integer $n$ such that $[x^k y^k - (xy)^k, x] = 0 = [x^k y^k - (xy)^k, y]$ $(k = n, n+1, n+2)$. Then, we get $[(xy)^k, yx] = 0$ $(k = n, n+1)$. The last equalities yield $[xy, yx] = 0$. Hence $R$ is commutative by Lemma 4.

We now proceed to the general case. Let $x, y \in J$, and $e$ a pseudo-identity of $\{x, y\}$. Then, by making use of the argument employed above, we can easily see that $[(e+x)(e+y), (e+y)(e+x)] = 0$, namely $[x+y+xy, x+y+yx] = 0$. Hence, $J$ is commutative by Lemma 4. Since $D \subseteq J$ by the first step, $D$ is a commutative ideal, and therefore $D \subseteq C$ by [5, Lemma 1. 5]. Again by the same lemma, we obtain eventually $R = C$.

3) $\Longrightarrow$ 1). The proof is quite similar to that of 2) $\Longrightarrow$ 1).

4) $\Longrightarrow$ 1). Without loss of generality, we may assume that $R$ is prime. As is easily seen, $[(xy)^n, (yx)^m] = 0$ and $[(xy)^{n+l}, (yx)^m] = 0$ yield

$$(*) \qquad (xy)^n [(xy)^l, (yx)^m] = 0.$$

Now, let $a^2=0$. Let $u, v$ be arbitrary elements of the nil right ideal $aR$ (Lemma 3), and $e$ a pseudo-identity of $\{u, v\}$. Then, by (*), we readily obtain $[\{(e+u)(e+v)\}^l, \{(e+v)(e+u)\}^m]=0$. This means that $aR$ is a P. I. -ring, and then [6, Lemma 2. 1. 1] shows that $a=0$; $R$ is a reduced ring. As is well known, the prime reduced ring $R$ has no non-zero zero-divisors. Consequently, again by (*), we get $[(xy)^l, (yx)^m]=0$. Hence, $R$ is commutative by [8, Theorem 3].

**3.** Finally, we prove the following commutativity theorem for torsion-free $s$-unital semi-prime rings.

**Theorem 3.** *Let $R$ be a torsion-free s-unital semi-prime ring. If for each $x, y \in R$ there exists a positive integer $n=n(x, y)$ such that $[(xy)^n-(yx)^n, x]=0$, then $R$ is commutative.*

In preparation for the proof of Theorem 3, we establish the following lemmas.

**Lemma 5.** *Let $R$ be an s-unital ring in which for each pair of elements $x, y$ there exists a positive integer $n=n(x, y)$ such that $[(xy)^n-(yx)^n, x]=0$. If $a$ is quasi-regular, then for each $x \in R$ there exist positive integers $k$, $l$ such that $[a, [a, x^k]]=0$ and $l!\,[a, x^k]^l=0$.*

*Proof.* Let $e$ be a pseudo-identity of $\{a, x\}$, and $a'$ the quasi-inverse of $a$. Using the hypothesis for the elements $e+a$ and $x(e+a')$, we can find a positive integer $k$ such that $[x^k-(e+a)x^k(e+a'), e+a]=0$. Then $[a, [a, x^k]]=[x^k-(e+a)x^k(e+a'), e+a](e+a)=0$, and we can apply the argument employed in the last part of the proof of Lemma 1.

**Lemma 6.** *Let $R$ be a torsion-free s-unital semi-prime ring in which for each pair of elements $x, y$ there exists a positive integer $n=n(x, y)$ such that $[(xy)^n-(yx)^n, x]=0$. Then $R$ is a reduced ring.*

*Proof.* Let $a^2=0$. Then $aR$ is a nil right ideal (see the proof of Lemma 2). Let $u, v \in aR$, and $e$ a pseudo-identity of $\{u, v\}$. By Lemma 5, there exists a positive integer $k$ such that $[u, [u, (e+v)^k]]=0$. If $v^2=0$, then $k[u, [u, v]]=[u, [u, (e+v)^k]]=0$. Hence $[u, [u, v]]=0$. Now, by making use of induction on the nilpotency indices of nilpotent elements $v$, we can easily see that $[u, [u, v]]=0$ for all $u, v \in aR$. Thus, [6, Lemma 2. 1. 1] shows that $a=0$.

*Proof of Theorem* 3. In view of Lemmas 5 and 6, we can proceed in the same way as that of Theorem 1.

## REFERENCES


[1] H. ABU-KHUZAM: On rings with nil commutator ideals, Bull Austral. Math. Soc. **23** (1981), 307—311.

[2] H.E. BELL: On some commutativity theorems of Herstein, Archiv Math. **24** (1973), 34—38.

[3] L.P. BELLUCE, I.N. HERSTEIN and S.K. JAIN: Generalized commutative rings, Nagoya Math. J. **27** (1966), 1—5.

[4] V. GUPTA: A commutativity theorem for semiprime rings, J. Austral. Math. Soc. **30 A** (1980), 33—36.

[5] I.N. HERSTEIN: Topics in Ring Theory, Univ. Chicago Press, Chicago, 1969.

[6] I.N. HERSTEIN: Rings with Involution, Univ. Chicago Press, Chicago, 1976.

[7] I.N. HERSTEIN: A commutativity theorem, J. Algebra **38** (1976), 112—118.

[8] M. HONGAN and H. TOMINAGA: Some commutativity theorems for semi-prime rings, Hokkaido Math. J., to appear.

[9] I. MOGAMI and M. HONGAN: Note on commutativity of rings, Math. J. Okayama Univ. **20** (1978), 21—24.



OKAYAMA UNIVERSITY

TSUYAMA COLLEGE OF TECHNOLOGY

OKAYAMA UNIVERSITY


(*Received October 31, 1980*)