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A NOTE ON ISOMORPHISM INVARIANTS OF A MODULAR GROUP ALGEBRA

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1. Introduction. Let $F(G)$ be the group algebra of a group G over the prime field $F=\mathrm{GF}(p)$ and let $\{M_{i,p}(G)\}_{i\geqq 1}$ be the Brauer-Jennings-Zassenhaus M-series of G relative to the prime p: $M_{1,p}(G)=G$ and $M_{i,p}(G)=(G, M_{i-1,p}(G))M_{(i/p),p}(G)^p$ for $i\geqq 2$, where (i/p) is the least integer not smaller than i/p and $(G, M_{i-1,p}(G))$ is the subgroup generated by all commutators $(x,y)=x^{-1}y^{-1}xy$, $x\in G$, $y\in M_{i-1,p}(G)$. In [4], I. B. S. Passi and S. K. Sehgal showed that for each $i\geqq 1$ the factor groups $M_{i,p}(G)/M_{i+1,p}(G)$ and $M_{i,p}(G)/M_{i+2,p}(G)$ are isomorphism invariants of $F(G)$. In this note we shall show that the factor groups $M_{i,p}(G)/M_{i+j,p}(G)$ are isomorphism invariants of $F(G)$ for all $i\geqq 1$ and all j with $1\leqq j\leqq i+1$, too.

2. Notations and preliminary results. Let G be a group, N a normal subgroup of G, and R a commutative ring with identity. We adopt the following notations:

$R(G)$ = the group ring of G with coefficients in R.
$\Delta_R(G,N)$ = the kernel of the natural homomorphism $R(G)\to R(G/N)$.
$\Delta_R(G)=\Delta_R(G,G)$ (the augmentation ideal of $R(G)$).
$\Delta_R^i(G)$ = the i-th power of $\Delta_R(G)$.
$U(R(G))$ = the unit group of $R(G)$.

It is easy to verify that if S and I are subrings of a ring such that $SI+IS\subseteqq I$ then $S+I$ is a subring which contains I as an ideal. Now, let G^* be a subgroup of G, and I an R-submodule of $R(G)$ satisfying $I^2\subseteqq I$ and $\Delta_R(G^*)I+I\Delta_R(G^*)\subseteqq I$. Since $R(G^*)=\Delta_R(G^*)+R$, we see that $R(G^*)+I$ forms a ring containing I as an ideal. Let

$$\nu:\ U(R(G^*))\to U(R(G^*)+I/I);\ u\longrightarrow u+I$$

be the group homomorphism induced by the natural ring homomorphism $R(G^*)\to R(G^*)+I/I$. Denoting by ν^* the restriction of ν to G^*, we see that the kernel of ν^* coincides with $G^*\cap(1+I)$ and the image of ν^* is G^*+I/I. Hence, we have an isomorphism $G^*/G^*\cap(1+I)\cong G^*+I/I$. The next is an immediate consequence of this fact.

Lemma 1. *Let* $\theta: R(G) \to R(H)$ *be an R-algebra isomorphism. Let* G^* *and* H^* *be subgroups of* G *and* H *respectively, and* I *an R-submodule of* $R(G)$ *such that* $I^2 \subseteqq I$, $\Delta_R(G^*)I + I\Delta_R(G^*) \subseteqq I$ and $\theta(G^* + I) = H^* + \theta(I)$, *then* $G^*/G^* \cap (1 + I) \cong H^*/H^* \cap (1 + \theta(I))$.

Let F be the prime field $\mathrm{GF}(p)$. Then, it is known that for each $i \geqq 1$, $M_{i,p}(G)$ coincides with $D_{i,F}(G) = G \cap (1 + \Delta_F^i(G))$, the i-th dimension subgroup of G over F (see, e. g. [1, 2, 3, 5 and 6]). Now, let $L_{i,p}(G) = \Delta_F(G, M_{i,p}(G)) + \Delta_F^{i+1}(G)$ for $i \geqq 1$.

We borrow the following in [4].

Lemma 2. (1) $L_{i,p}(G) = \{x - 1 + \alpha \mid x \in M_{i,p}(G), \quad \alpha \in \Delta_F^{i+1}(G)\}$ *for* $i \geqq 1$.

(2) *Let* $\theta: F(G) \to F(H)$ *be a normalized isomorphism in the sense that the sum of the coefficients of* $\theta(g)$ *is* 1 *for all* $g \in G$. *Then* $\theta(L_{i,p}(G)) = L_{i,p}(H)$ *for all* $i \geqq 1$.

3. Main theorem. We are now in a position to prove our main theorem.

Theorem. *Let* F *be the prime field* $\mathrm{GF}(p)$, *and* $\{M_{i,p}(G)\}_{i\geqq 1}$ *the Brauer-Jennings-Zassenhaus M-series of* G *relative to the prime* p. *If* $F(G) \cong F(H)$, *then* $M_{i,p}(G)/M_{i+j,p}(G) \cong M_{i,p}(H)/M_{i+j,p}(H)$ *for all* $i \geqq 1$ *and all* j *with* $1 \leqq i \leqq j + 1$.

Proof. Throughout the proof, we shall omit their subscripts p and F from $M_{i,p}(\ )$, $L_{i,p}(\ )$ and $\Delta_F(\ )$, which are denoted by $M_i(\ )$, $L_i(\ )$ and $\Delta(\ )$, respectively.

Let $I_{i,i} = L_i(G)$, and $I_{i,i+1} = \Delta^{i+1}(G)$ $(i \geqq 1)$. Since

$$I_{i,i+1} \supseteqq I_{i+1,i+1} \supseteqq I_{i+1,i+2} \qquad (i \geqq 1)\,,$$

we can find subspaces $I_{i,i+2}$ of $I_{i,i+1}$ containing $I_{i+1,i+2}$ such that $I_{i,i+1} = I_{i+1,i+1} + I_{i,i+2}$ and $I_{i+1,i+2} \supseteqq I_{i+1,i+1} \cap I_{i,i+2}$. Obviously,

$$I_{i,i+2} \supseteqq I_{i+1,i+2} \supseteqq I_{i+1,i+3} \qquad (i \geqq 1)\,,$$

and so we can repeat the same procedure to obtain subspaces $I_{i,i+3}$ of $I_{i,i+2}$ containing $I_{i+1,i+3}$ such that $I_{i,i+2} = I_{i+1,i+2} + I_{i,i+3}$ and $I_{i+1,i+3} \supseteqq I_{i+1,i+2} \cap I_{i,i+3}$. In this way, for $j \geqq 0$ we can construct inductively the series $\{I_{i,i+j}\}_{i\geqq 1}$ of subspaces of $F(G)$ such that

(1) $I_{i,i+j} \supseteqq I_{i,i+j+1} \supseteqq I_{i+1,i+1+j}$ $\qquad (i \geqq 1\,;\ j \geqq 0)$

(2) $I_{i,i+j} = I_{i+1,i+j} + I_{i,i+j+1}$ $\qquad (i \geqq 1;\ j \geqq 1)$

(3) $I_{i+1,i+1+j} \supseteqq I_{i+1,i+j} \cap I_{i,i+j+1}$ $\qquad (i \geqq 1;\ j \geqq 1)$.

From (1), we see that $\{I_{i,i+j}\}_{j\geqq 0}$ is a decreasing series for $i \geqq 1$, and moreover if $1 \leqq i \leqq k$ then $I_{i,k+1} \supseteqq I_{i+1,k+1}$. We have therefore

(4) $I_{1,k+1} \supseteqq I_{2,k+1} \supseteqq \cdots \supseteqq I_{k,k+1} \supseteqq I_{k+1,k+1}$ $\qquad (k \geqq 1)$.

Similarly, by (2) and (3), we can prove

(5) $I_{i,k+1} = I_{i+1,k+1} + I_{i,k+2}$ $\qquad (1 \leqq i \leqq k)$

(6) $I_{i+1,k+2} \supseteqq I_{i+1,k+1} \cap I_{i,k+2}$ $\qquad (1 \leqq i \leqq k)$.

Combining (4) and (5), we obtain

(7) $I_{i,k+1} = I_{k+1,k+1} + I_{i,k+2}$ $\qquad (1 \leqq i \leqq k)$.

Since $I_{k+1,k+1} = \{x - 1 + \alpha \mid x \in M_{k+1}(G),\ \alpha \in I_{k+1,k+2}\}$ by Lemma 2 (1), (7) together with (1) and (4) implies

(8) $I_{i,i+j} = \{x - 1 + \alpha \mid x \in M_{i+j}(G),\ \alpha \in I_{i,i+j+1}\}$ $\qquad (i \geqq 1;\ j \geqq 0)$.

Now, we claim that

(9) $I_{i,i} = \{x - 1 + \alpha \mid x \in M_i(G),\ \alpha \in I_{i,i+j+1}\}$ $\qquad (i \geqq 1;\ j \geqq 0)$.

According to (1), it suffices to show that the left-hand side of (9) is contained in the right-hand side. We shall proceed by induction on j, keeping i fixed. The first step of induction, when $j = 0$, is assured by (8). Suppose $j \geqq 1$ and the statement holds for $j - 1$. Given $\beta \in I_{i,i}$, by the induction hypothesis, $\beta = x - 1 + \alpha$ with some $x \in M_i(G)$ and $\alpha \in I_{i,i+j}$. By (8), $\alpha = y - 1 + \gamma$ with some $y \in M_{i+j}(G)$ and $\gamma \in I_{i,i+j+1}$. Therefore,

$$\begin{aligned}\beta &= x - 1 + y - 1 + \gamma \\ &= xy - 1 + \delta, \qquad \text{where } \delta = \gamma - (x-1)(y-1).\end{aligned}$$

By (4),

$$(x-1)(y-1) \in \Delta^i(G)\Delta^{i+j}(G) \subseteqq \Delta^{i+j+1}(G) = I_{i+j,i+j+1} \subseteqq I_{i,i+j+1},$$

which implies $\delta \in I_{i,i+j+1}$. Since $xy \in M_i(G)$, the induction is complete and hence (9) is established.

Next, we claim that

(10) $$G \cap (1 + I_{i,k+1}) = M_{k+1}(G) \qquad (1 \leqq i \leqq k).$$

By (4), the right-hand side of (10) is contained in the left-hand side. To show the reverse inclusion, we proceed by induction on k, the statement being clear for $k = 1$. Suppose $G \cap (1 + I_{i,k+1}) \subseteqq M_{k+1}(G)$ $(1 \leqq i \leqq k)$. To complete the induction, we have to show that

$$G \cap (1 + I_{t,k+2}) \subseteqq M_{k+2}(G) \qquad (1 \leqq t \leqq k+1).$$

To see this, we use descending induction on t, the above being obvious for $t = k+1$. Assume that $G \cap (1 + I_{t,k+2}) \subseteqq M_{k+2}(G)$ for some t with $2 \leqq t \leqq k+1$. Then, by our induction hypothesis $G \cap (1 + I_{t-1,k+1}) \subseteqq M_{k+1}(G)$. Let g be in $G \cap (1 + I_{t-1,k+2}) (\subseteqq G \cap (1 + I_{t-1,k+1})$ by (1)). Then, $g \in M_{k+1}(G)$, and hence $g - 1 \in L_{k+1}(G) = I_{k+1,k+1} \subseteqq I_{t,k+1}$ by (4). Noting here that $I_{t,k+1} \cap I_{t-1,k+2} \subseteqq I_{t,k+2}$ by (6), we obtain $g - 1 \in I_{t,k+2}$. Now, according to the decreasing induction hypothesis, it follows $g \in M_{k+2}(G)$. This completes the induction on k, and hence (10) has been proved.

Now, assume that an isomorphism $\theta: F(G) \to F(H)$ is given. Then, without loss of generality, we may assume that θ is normalized, and therefore $\theta(\Delta^i(G)) = \Delta^i(H)$ and $\theta(L_i(G)) = L_i(H)$ (Lemma 2 (2)). Hence, applying the above argument to the subspaces $\theta(I_{i,i+j})$ of $F(H)$, we do have the following:

(9′) $\theta(I_{i,i}) = \{h - 1 + \beta \mid h \in M_i(H),\ \beta \in \theta(I_{i,i+j+1})\}$ $(i \geqq 1;\ j \geqq 0)$.

(10′) $H \cap (1 + \theta(I_{i,k+1})) = M_{k+1}(H)$ $(1 \leqq i \leqq k)$.

(9) and (9′) immediately imply

(11) $\theta(M_i(G) + I_{i,i+j+1}) = M_i(H) + \theta(I_{i,i+j+1})$ $(i \geqq 1;\ j \geqq 0)$.

We are now ready to complete the proof of our theorem. Let i, j satisfy $i \geqq 1$ and $1 \leqq j \leqq i+1$. Then, since $I_{i+j-1,i+j} \subseteqq I_{i,i+j} \subseteqq I_{i,i+1} = \Delta^{i+1}(G)$ by (1) and (4), there holds that

$$I^2_{i,i+j} \subseteqq \Delta^{2i+1}(G) \subseteqq \Delta^{i+j}(G) = I_{i+j-1,i+j} \subseteqq I_{i,i+j}.$$

Similarly,

$$\Delta(M_i(G)) I_{i,i+j} + I_{i,i+j} \Delta(M_i(G)) \subseteqq I_{i,i+j}.$$

Finally, by (11)

$$\theta(M_i(G) + I_{i,i+j}) = M_i(H) + \theta(I_{i,i+j}).$$

Thus, in virtue of Lemma 1, we get

$M_i(G)/M_i(G) \cap (1 + I_{i,i+j}) \cong M_i(H)/M_i(H) \cap (1 + \theta(I_{i,i+j}))$. Since $M_i(G) \cap (1 + I_{i,i+j}) = M_{i+j}(G)$ and $M_i(H) \cap (1 + \theta(I_{i,i+j})) = M_{i+j}(H)$ by (10) and (10′), the theorem has been proved.

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