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J-GROUPS OF THE ORBIT MANIFOLDS $(S^{2m+1}\times S^l)/D_n$ BY THE DIHEDRAL GROUP D_n

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Introduction. Let n ($\geqq 3$) be an odd integer, and D_n the dihedral group of order $2n$. Let S^{2m+1} (resp. S^l) be the unit sphere in C^{m+1} (resp. R^{l+1}). Let $D_n(m, l)$ be the orbit manifold $(S^{2m+1}\times S^l)/D_n$ (see §1). The K-ring of $D_n(m, l)$ has been studied by Imaoka and Sugawara [8]. The purpose of this paper is to calculate the J-group $\widetilde{J}(D_n(m, l))$ for odd prime n. The main theorem of §1 will give the direct sum decomposition of $\widetilde{KO}(D_n(m, l))$ (Theorem 1. 12). The direct sum decomposition of $\widetilde{J}(D_n(m, l))$ will be given in §2 (Theorem 2. 4), and the direct summands of $\widetilde{J}(D_n(m, l))$ will be discussed in §3.

1. Preliminaries and decompositions of $\widetilde{KO}(D_n(m, l))$. Let n ($\geqq 3$) be an odd integer and D_n the dihedral group of order $2n$ generated by two elements g and t with relations $g^n = t^2 = gtgt = 1$. Let S^{2m+1} and S^l be the unit spheres in the complex $(m+1)$-space C^{m+1} and the real $(l+1)$-space R^{l+1} respectively. Then D_n operates freely on the product space $S^{2m+1}\times S^l$ by

$$g\cdot(z, x) = (z\exp(2\pi\sqrt{-1}/n), x)$$
$$t\cdot(z, x) = (\bar{z}, -x) \qquad (z\in S^{2m+1}, x\in S^l),$$

where $\bar{z}$ is the conjugate of z. Then we have the orbit manifold

$$D_n(m, l) = (S^{2m+1}\times S^l)/D_n = (L^m(n)\times S^l)/Z_2,$$

where $L^m(n) = S^{2m+1}/Z_n$ is the standard lens space, and the action of Z_2 is given by

$$t\cdot(([z], x) = ([\bar{z}], -x) \qquad ([z]\in L^m(n), x\in S^l).$$

The lens space $L^m(n)$ has the cell decompsition

$$L^m(n) = C^0\cup C^1\cup\cdots\cup C^{2m}\cup C^{2m+1},\ \partial(C^{2i+1}) = 0,\ \partial(C^{2i}) = nC^{2i-1},$$

which is invariant under the conjugation. Also, S^l has the cell decomposition

$$S^l = D^0_+\cup D^0_-\cup D^1_+\cup D^1_-\cup\cdots\cup D^l_+\cup D^l_-$$

such that $S^j = \overline{D}^j_+\cup\overline{D}^j_-\supset\overline{D}^j_+\cap\overline{D}^j_- = S^{j-1}$. Let $\pi: L^m(n)\times S^l\longrightarrow D_n(m, l)$ be the projection. Then it is known that $D_n(m, l)$ is the cell complex with cells defined by

$$(C^i,\ D^j)=\pi(C^i\times D^j_+)\qquad(0\leqq i\leqq 2m+1,\ 0\leqq j\leqq l),$$

which have the boundary operations

$$\partial(C^{2i+1},\ D^j)=((-1)^i+(-1)^{j+1})(C^{2i+1},\ D^{j-1})$$
$$\partial(C^{2i},\ D^j)=n(C^{2i-1},\ D^j)+((-1)^i+(-1)^j)(C^{2i},\ D^{j-1})$$

(cf. [9]). Consider the $2m$-skeleton

$$L^m_0(n)=C^0\cup C^1\cup\cdots\cup C^{2m}$$

of $L^m(n)$, and the subcomplex

$$D^0_n(m,\ l)=(L^m_0(n)\times S^l)/Z_2$$

of $D_n(m,\ l)$ with cells $\{(C^i,\ D^j)\mid 0\leqq i\leqq 2m,\ 0\leqq j\leqq l\}$, and identify the real l-dimensional projective space $RP(l)$ with the subcomplex $D^0_n(0,\ l)$ of $D^0_n(m,\ l)$. Denote by $(c^i,\ d^j)$ the dual cochain of $(C^i,\ D^j)$. Then we have the following

Lemma 1.1 ([8, Lemma 1.8]).

(1) $H^*(D^0_n(m,\ l),\ RP(l))$

$$\cong\begin{cases}\sum_{i=1}^{[m/2]} Z_n(c^{4i},\ d^0)\oplus\sum_{i=1}^{[(m+1)/2]} Z_n(c^{4i-2},\ d^l) & (l:\ even)\\ \sum_{i=1}^{[m/2]} Z_n(c^{4i},\ d^0)\oplus\sum_{i=1}^{[m/2]} Z_n(c^{4i},\ d^l) & (l:\ odd),\end{cases}$$

where $Z_n(c^i,\ d^j)$ *means the cyclic group of order* n *generated by* $(c^i,\ d^j)$.

(2) $H^*(D^0_n(m,\ l),\ RP(l);\ Z_2)=0.$

The following lemma can be obtained by making use of Lemma 1. 1 and the Atiyah-Hirzebruch spectral sequence for KO-theory.

Lemma 1.2. *The order of* $\widetilde{KO}^i(D^0_n(m,\ l)/RP(l))$ *is a divisor of* n^m. *Especially,*

$$(1)\quad \mathrm{ord}\ \widetilde{KO}(D^0_n(m,\ l)/RP(l))=\begin{cases}n^m & (l\equiv 2\ (\mathrm{mod}\ 4))\\ n^{[m/2]} & (l\equiv 0\ (\mathrm{mod}\ 4))\\ \text{a divisor of } n^{[m/2]} & (l:\ \mathrm{odd}),\end{cases}$$

$$(2)\quad \mathrm{ord}\ \widetilde{KO}^{-1}(D^0_n(m,\ l)/RP(l))=\begin{cases}0 & (l\not\equiv 3\ (\mathrm{mod}\ 4))\\ \text{a divisor of } n^{[m/2]} & (l\equiv 3\ (\mathrm{mod}\ 4)),\end{cases}$$

where ord G *means the order of a finite group* G.

We consider the following maps

$$\begin{array}{llll} & i: L^m(n) \longrightarrow D_n(m, l), & & i_0: L_0^m(n) \longrightarrow D_n^0(m, l), \\ (1.3) & k: RP(l) \longrightarrow D_n(m, l), & & j: D_n^0(m, l) \longrightarrow D_n(m, l), \\ & p: D_n(m, l) \longrightarrow RP(l), & & q_1: D_n(m, l) \longrightarrow D_n(m, l)/D_n^0(m, l), \end{array}$$

where $i([z])=[[z],(1, 0, \cdots, 0)]$, $k([x]) = [[(1, 0, \cdots, 0)], x]$, $p([[z], x]) = [x]$, j is the inclusion map, q_1 is the quotient map and i_0 is the restriction of i.

It is known that there is a homeomorphism

$$(1.4) \qquad f: D_n(m, l)/D_n^0(m, l) \longrightarrow S^m \wedge (RP(m+l+1)/RP(m)),$$

where the right hand term is the suspension of the stunted real projective space (cf. [8, Lemma 1.12]). The next proposition is shown in [6].

Proposition 1.5. *The order of the torsion part of the group* $\widetilde{KO}^i(RP(m+l+1)/RP(m))$ *is a power of 2. Especially, the groups* $\widetilde{KO}(S^m \wedge (RP(m+l+1)/RP(m)))$ *are tabled as follows, where* (t) *is a cyclic group of order* t, *and* $\phi(n_1, n_2)$ *is the number of integers* s *with* $n_2 < s \leqq n_1$ *and* $s \equiv 0, 1, 2$ *or* 4 (mod 8).

m (mod 4) \ l (mod 8)	0	1	2	3	4	5	6	7
0	$(2^{\phi(2m+l+1,\,2m)})$							
1	0	(∞)	0	0	0	(∞)	0	0
2	0	0	0	(2)	$(2)\oplus(2)$	(2)	0	0
3	0	(∞)	(2)	$(2)\oplus(2)$	(2)	(∞)	0	0

By making use of Lemma 1.2, Proposition 1.5 and the fact $p\circ k = 1_{RP(l)}$, we can easily obtain

Lemma 1.6. *There is a commutative diagram*

$$(1.7)\qquad \begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & \widetilde{KO}(S^m \wedge (RP(m+l+1)/RP(m))) & = & \widetilde{KO}(S^m \wedge (RP(m+l+1)/RP(m))) & & \\ & & \downarrow q_1^! f^! & & \downarrow q_1^! f^! & & \\ 0 \to & & \widetilde{KO}(D_n(m, l)/RP(l)) & \longrightarrow & \widetilde{KO}(D^n(m, l)) & \xrightarrow{k^!} & \widetilde{KO}(RP(l)) \to 0 \\ & & \downarrow j^! & & \downarrow j^! & & \| \\ 0 \to & & \widetilde{KO}(D_n^0(m, l)/RP(l)) & \longrightarrow & \widetilde{KO}(D_n^0(m, l)) & \xrightarrow{k^!} & \widetilde{KO}(RP(l)) \to 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

of exact sequences. Especially, the rows are split exact, and $j^!$: $\widetilde{KO}(D_n(m, l)) \longrightarrow \widetilde{KO}(D_n^0(m, l))$ *is monomorphic on odd torsion.*

Considering the D_n-action on $S^{2m+1} \times S^l \times C$ given by

$$t \cdot (z, x, u) = (\bar{z}, -x, \bar{u})$$
$$g \cdot (z, x, u) = (z \exp(2\pi\sqrt{-1}/n), x, u \exp(2\pi\sqrt{-1}/n))$$

for $(z, x, u) \in S^{2m+1} \times S^l \times C$, we have a real 2-plane bundle

$$\eta_1 : (S^{2m+1} \times S^l \times C)/D_n \longrightarrow D_n(m, l).$$

Denote by ξ the canonical real line bundle over $RP(l)$, and $\xi_1 = p^*\xi$ the induced bundle of ξ by the projection $p: D_n(m, l) \longrightarrow RP(l)$ in (1.3); by η the canonical complex line bundle over $L^m(n)$. Then we have the following elements:

$$\begin{array}{lll} & \lambda = \xi - 1 \in \widetilde{KO}(RP(l)), & \sigma = \eta - 1 \in \widetilde{K}(L^m(n)), \\ (1.8) & \sigma = j^!(\sigma) \in \widetilde{K}(L_0^m(n)), & \bar{\sigma} = r(\sigma) \in \widetilde{KO}(L_0^m(n)), \\ & \alpha_0 = \eta_1 - \xi_1 - 1 \in \widetilde{KO}(D_n(m, l)), & \alpha_0 = j^!(\alpha_0) \in \widetilde{KO}(D_n^0(m, l)), \end{array}$$

where r is the real restriction. Since $i^*\xi_1 = 1$, $i^*\eta_1 = r\eta$, $k^*\xi_1 = \xi$ and $k^*\eta_1 = \xi + 1$, we have the the following

Lemma 1.9. $i_0^!(\alpha_0) = \bar{\sigma}$, $k^!(\alpha_0) = 0$.

By definition, we readily see

Lemma 1.10. *The elements* α_0 *of* (1.8) *are natural with respect to the inclusions* $D_n(m', l') \subset D_n^0(m, l') \subset D_n(m, l)$ *for* $m' < m$, $l' \leqq l$.

Let

$$(1.11) \qquad \mathfrak{A}_{m,l} \subset \widetilde{KO}(D_n(m, l)), \quad \mathfrak{A}_{m,l,0} \subset \widetilde{KO}(D_n^0(m, l))$$

be the subrings generated by α_0 of (1.8). Then we have

Lemma 1.12. $\mathfrak{A}_{m,l}$ *is isomorphic to* $\mathfrak{A}_{m,l,0}$ *by* $j^!$: $\widetilde{KO}(D_n(m, l)) \longrightarrow \widetilde{KO}(D_n^0(m, l))$ *and* $\mathfrak{A}_{m,l,0}$ *is isomorphic to* $\widetilde{KO}(L_0^m(n))$. *And their orders are* $n^{[m/2]}$.

Proof. Assume that $l \not\equiv 2 \pmod 4$, and consider the diagram (1.7). In the lower exact row of the diagram $k^!(\alpha_0) = 0$ by Lemma 1.9. Hence ord $\mathfrak{A}_{m,l,0}$ is a divisor of $n^{[m/2]}$ by Lemma 1.2(1). Therefore, since $\mathfrak{A}_{m,l}$ is the image of $\mathfrak{A}_{m+1,l,0}$ by Lemma 1.10, ord $\mathfrak{A}_{m,l}$ is a divisor of $n^{[(m+1)/2]}$.

Then Lemma 1. 6 implies $\mathfrak{A}_{m,l} \cong \mathfrak{A}_{m,l,0}$.

Now consider the homomorphism

$$i_0^!: \widetilde{KO}(D_n^0(m, l)) \longrightarrow \widetilde{KO}(L_0^m(n)).$$

The ring $\widetilde{KO}(L_0^m(n))$ is generated by $\bar{\sigma}$ of (1. 8) and contains exactly $n^{[m/2]}$ elements (cf. [12, Proposition 2. 11]). By Lemma 1. 9 we have $i_0^!(\alpha_0) = \bar{\sigma}$ Therefore $\mathfrak{A}_{m,l,0}$ is isomorphic to $\widetilde{KO}(L_0^m(n))$ by $i_0^!$. Similarly we can prove the case $l \equiv 2 \pmod 4$. q. e. d.

The following result is immediate by Lemmas 1. 2, 1. 6 and 1. 12.

Proposition 1.13. *Suppose that* $l \not\equiv 2 \pmod 4$. *Then we have the direct sum decompositions*

(1) $\widetilde{KO}(D_n^0(m, l)) = \mathfrak{A}_{m,l,0} \oplus p^!(\widetilde{KO}(RP(l)))$,

(2) $\widetilde{KO}(D_n(m, l)) = \mathfrak{A}_{m,l} \oplus q_1^! f^!(\widetilde{KO}(S^m \wedge (RP(m+l+1)/RP(m))))$
$\oplus p^!(\widetilde{KO}(RP(l)))$.

The projection $\pi: L^m(n) \times S^l \longrightarrow D_n(m, l)$ induces naturally the homeomorphism

$$\begin{aligned} h:\ & D_n(m, l)/(D_n(m, l-1) \cup RP(l)) \\ & \approx (L^m(n) \times \bar{D}_+^l)/(L^m(n) \times S^{l-1} \cup * \times \bar{D}_+^l) \\ & \approx (L^m(n) \times S^l)/(L^m(n) \times * \cup * \times S^l) \\ & = S^l \wedge L^m(n). \end{aligned}$$

The restriction of h

$$h_0: D_n^0(m, l)/(D_n^0(m, l-1) \cup RP(l)) \longrightarrow S^l \wedge L_0^m(n)$$

is also a homeomorphism.

We consider the homomorphisms

$$(1.14)\qquad \begin{aligned} \widetilde{K}(S^l \wedge L^m(n)) & \xrightarrow{r} \widetilde{KO}(S^l \wedge L^m(n)) \\ & \xrightarrow{h^!} \widetilde{KO}(D_n(m, l)/(D_n(m, l-1) \cup RP(l))) \\ & \xrightarrow{q^!} \widetilde{KO}(D_n(m, l)), \end{aligned}$$

where $q: D_n(m, l) \longrightarrow D_n(m, l)/(D_n(m, l-1) \cup RP(l))$ is the natural projection. Let

$$(1.15)\qquad \mathfrak{B}_{m,2l} \subset \widetilde{KO}(D_n(m, 2l))$$

be the image of $\widetilde{K}(S^{2l} \wedge L^m(n))$ by $q^! h^! r$.

Consider the following exact and commutative diagram:

$$\begin{array}{ccccccc} & & \widetilde{K}(S^{2l}\wedge L^m(n)) & \xrightarrow{j_C^!} & \widetilde{K}(S^{2l}\wedge L_0^m(n)) & \longrightarrow & 0 \\ & & \downarrow r & & \downarrow r & & \\ 0\longrightarrow \widetilde{KO}(S^{2m+2l+1}) & \longrightarrow & \widetilde{KO}(S^{2l}\wedge L^m(n)) & \xrightarrow{j^!} & \widetilde{KO}(S^{2l}\wedge L_0^m(n)) & \longrightarrow & 0, \end{array}$$

where $j: S^{2l}\wedge L_0^m(n) \longrightarrow S^{2l}\wedge L^m(n)$ is the inclusion map and $j_C^!$ is an isomorphism (cf. [12, Lemma 2. 4]). Since $\widetilde{KO}(S^{2l}\wedge L_0^m(n))$ and $\widetilde{K}(S^{2l}\wedge L_0^m(n))$ are of odd orders and $\widetilde{KO}(S^{2m+2l+1})$ has no odd torsion, there exists a splitting

$$\iota: \widetilde{KO}(S^{2l}\wedge L_0^m(n)) \longrightarrow \widetilde{KO}(S^{2l}\wedge L^m(n)),$$

which maps $\widetilde{KO}(S^{2l}\wedge L_0^m(n))$ isomorphically onto $r(\widetilde{K}(S^{2l}\wedge L_0^m(n)))$. We consider further

$$\nu = q^! h^! \iota. \tag{1.16}$$

Then we obtain

Proposition 1.17. (1) *$\widetilde{KO}(S^{4l+2}\wedge L_0^m(n))$ is mapped isomorphically onto $\mathfrak{B}_{m,4l+2}$ by ν.*

(2) $\widetilde{KO}(D_n(m, 4l+2)) = \mathfrak{A}_{m,4l+2}\oplus\mathfrak{B}_{m,4l+2}\oplus p^!(\widetilde{KO}(RP(4l+2)))$
$\oplus q_1^! f^!(\widetilde{KO}(S^m\wedge (RP(m+4l+3)/RP(m))))$.

Proof. The exact sequence of the triple $(D_n^0(m, 4l+2),\ D_n^0(m, 4l+1)\cup RP(4l+2),\ RP(4l+2))$ becomes

$$\begin{aligned} &\widetilde{KO}^{-1}(D_n^0(m, 4l+1)/RP(4l+1)) \longrightarrow \widetilde{KO}(S^{4l+2}\wedge L_0^m(n)) \\ &\xrightarrow{i_1^!} \widetilde{KO}(D_n^0(m, 4l+2)/RP(4l+2)) \xrightarrow{i_2^!} \widetilde{KO}(D_n^0(m, 4l+1)/RP(4l+1)) \\ &\longrightarrow \widetilde{KO}^1(S^{4l+2}\wedge L_0^m(n)) = 0, \end{aligned}$$

in which $\widetilde{KO}^{-1}(D_n^0(m, 4l+1)/RP(4l+1)) = 0$ by Lemma 1.2 (2). Let $\bar{i}_0: L_0^m(n) \longrightarrow D_0^m(m, l)/RP(l)$ be the composition of i_0 in (1.3) and the quotient map $D_n^0(m, l) \longrightarrow D_n^0(m, l)/RP(l)$. Then, by the proof of Lemma 1.12, the induced homomorphism $\bar{i}_0^!: \widetilde{KO}(D_n^0(m, l)/RP(l)) \longrightarrow \widetilde{KO}(L_0^m(n))$ is an isomorphism for $l \not\equiv 2 \pmod 4$. Hence we see that $i_2^!$ has a right inverse. This implies $\widetilde{KO}(D_n^0(m, 4l+2)/RP(4l+2)) \cong \widetilde{KO}(S^{4l+2}\wedge L_0^m(n)) \oplus \widetilde{KO}(L_0^m(n))$.

Let q_0 be the restriction of q in (1.14), and consider the following commutative diagram

$$
\begin{array}{ccccc}
 & & & & 0 \\
 & & & & \downarrow \\
0 \longrightarrow \widetilde{KO}(S^{4l+2}\wedge L_0^m(n)) & \xrightarrow{i_1} & \widetilde{KO}(D_n^0(m,4l+2)/RP(4l+2)) & \xrightarrow{i_2} & \widetilde{KO}(D_n^0(m,4l+1)/RP(4l+1)) \longrightarrow 0 \\
\downarrow h_0^! & & \downarrow & & \downarrow \\
\widetilde{KO}(D_n^0(m,4l+2)/(D_n^0(m,4l+1)\cup RP(4l+2))) & \xrightarrow{q_0^!} & \widetilde{KO}(D_n^0(m,4l+2)) & \xrightarrow{i_0^!} & \widetilde{KO}(L_0^m(n)), \\
 & & \downarrow k^! & & \\
(1.18) & & \widetilde{KO}(RP(4l+2)) & & \\
 & & \downarrow & & \\
 & & 0 & &
\end{array}
$$

in which the upper row and the column are exact. Then $q_0^! h_0^!$ is monomorphic and (1) follows from the commutative diagram

$$
(1.19)\qquad
\begin{array}{ccccc}
\widetilde{K}(S^{4l+2}\wedge L^m(n)) & \xrightarrow{r} & \widetilde{KO}(S^{4l+2}\wedge L^m(n)) & \xrightarrow{q^! h^!} & \widetilde{KO}(D_n(m,4l+2))) \\
 & & \uparrow \iota & & \downarrow j^! \\
 & & \widetilde{KO}(S^{4l+2}\wedge L_0^m(n)) & \xrightarrow{q_0^! h_0^!} & \widetilde{KO}(D_n^0(m,4l+2)).
\end{array}
$$

Moreover the diagram (1.18) shows that

$$
\widetilde{KO}(D_n^0(m,4l+2)) \cong p_0^!(\widetilde{KO}(RP(4l+2)))\oplus \mathfrak{A}_{m,4l+2,0} \oplus q_0^! h_0^!(\widetilde{KO}(S^{4l+2}\wedge L_0^m(n))).
$$

Since $j^!: \mathfrak{B}_{m,4l+2} \cong q_0^! h_0^!(\widetilde{KO}(S^{4l+2}\wedge L_0^m(n)))$, (2) is an easy consequence of Lemmas 1.6 and 1.12. q. e. d.

Remark. Inspecting the diagrams which are similar to (1.18) and (1.19), we can see $\mathfrak{B}_{m,4l} = 0$.

By Lemma 1.12, we have a homomorphism

$$
(1.20)\qquad \mu:\ \widetilde{KO}(L_0^m(n)) \longrightarrow \widetilde{KO}(D_n(m,l))
$$

defined by $\mu(\bar{\sigma}) = \alpha_0$. Now, by Lemma 1.6 and Propositions 1.13, 1.17, we can see the following

Theorem 1.21. (1) *If* $l \not\equiv 2 \pmod 4$, *then the map*:

$$\theta: \widetilde{KO}(RP(l)) \oplus \widetilde{KO}(S^m \wedge (RP(m+l+1)/RP(m))) \oplus \widetilde{KO}(L_0^m(n)) \longrightarrow \widetilde{KO}(D^n(m, l))$$

defined by

$$\theta(x, y, z) = p^!(x) + q_1^! f^!(y) + \mu(z)$$

is an isomorphism.

(2) If $l \equiv 2 \pmod 4$, *then the map*

$$\theta: \widetilde{KO}(RP(l)) \oplus \widetilde{KO}(S^m \wedge (RP(m+l+1)/RP(m))) \oplus \widetilde{KO}(L_0^m(n)) \oplus \widetilde{KO}(S^l \wedge L_0^m(n)) \longrightarrow \widetilde{KO}(D_n(m, l))$$

defined by

$$\theta(x, y, z, w) = p^!(x) + q_1^! f^!(y) + \mu(z) + \nu(w)$$

is an isomorphism.

Remark. (1) The groups $\widetilde{KO}(RP(l))$ and $\widetilde{KO}(S^m \wedge (RP(m+l+1)/RP(m)))$ are known in [1] and [6]. The groups $\widetilde{KO}(L_0^m(n))$ and $\widetilde{KO}(S^l \wedge L_0^m(n))$ are known in [13].

(2) By definition, it is easy to see that the element $\alpha_0 \in \widetilde{KO}(D_n(m, l))$ in (1.8) corresponds to $\alpha \in \widetilde{K}(D_n(m, l))$ in [8, (1.13)] by the comlexification. Also, the ideal $B_{m,2l}$ *of* $\widetilde{K}(D_n(m, l))$ in [8, (2.23)] satisfies $r B(_{m,2l}) = \mathfrak{B}_{m,2l}$. In short, the direct sum decompositions in Theorem 1.21 and [8, Theorem 3.9] are compatible with the real restriction r and the complexification c.

2. Decompositions of $\widetilde{J}(D_n(m, l))$. In this section we recall from [2], [3] and [14] the basic properties of the J-groups for finite CW-complexes, and give direct sum decompositions of $\widetilde{J}(D_n(m, l))$.

A ψ-group is an abelian group Y together with given endomorphisms $\psi^k: Y \longrightarrow Y$ for each $k \in Z$. A ψ-map between ψ-groups is a homomorphism which commutes with the operations ψ^k. Let e be a function which assigns to each pair $k \in Z$, $y \in Y$ a non-negative integer $e(k, y)$. Then Y_e is defined to be the subgroup of Y generated by $\{k^{e(k,y)}(\psi^k - 1)y \mid k \in Z, y \in Y\}$: $Y_e = \langle \{k^{e(k,y)}(\psi^k - 1)y \mid k \in Z, y \in Y\} \rangle$. We now define

$$J''(Y) = Y / \bigcap_e Y_e,$$

where the intersection runs over all functions e (cf. [3, p. 144]).

If Y is a finite ψ-group, we have

$$J''(Y) = Y/\sum_k(\bigcap_e k^e(\psi^k - 1)Y), \tag{2.1}$$

where the intersection runs over all non-negative integers e.

Since a ψ-map $f: Y_1 \longrightarrow Y_2$ induces the homomorphism $\bar{f}: J''(Y_1) \longrightarrow J''(Y_2)$ (cf. [3, p. 145]), we can easily obtain

Lemma 2.2. *For any short exact sequence*

$$0 \longrightarrow Y_1 \xrightarrow{f} Y_2 \xrightarrow{g} Y_3 \longrightarrow 0 \tag{$*$}$$

of ψ-groups and ψ-maps, the following three statements are equivalent:

(1) *The ψ-map f has a left inverse.*

(2) *The ψ-map g has a right inverse.*

(3) *The short exact sequence $(*)$ splits. That is, the ψ-subgroup $f(Y_1)$ of Y_2 is a direct summand of Y_2.*

When this is the case, $(*)$ induces the split exact sequence

$$0 \longrightarrow J''(Y_1) \xrightarrow{\bar{f}} J''(Y_2) \xrightarrow{\bar{g}} J''(Y_3) \longrightarrow 0$$

of abelian groups and homomorphisms.

For each finite CW-complex X, $\widetilde{KO}(X)$ is a ψ-group by the Adams operations ψ^k. Denote by $\tilde{J}(X)$ the image of $\widetilde{KO}(X)$ by the homomorphism $J: KO(X) \longrightarrow J(X)$. According to Adams [2], [3] and Quillen [14], we have

$$\tilde{J}(X) \cong J''(\widetilde{KO}(X)). \tag{2.3}$$

We can check easily that the all splitting homomorphisms used in the proof of Theorem 1.21 are ψ-maps. Hence by making use of Lemma 2.2 and (2.3), we readily obtain the following theorem from Theorem 1.21.

Theorem 2.4. (1) *If* $l \not\equiv 2 \pmod 4$, *then the map*

$$\theta: \tilde{J}(RP(l)) \oplus \tilde{J}(S^m \wedge (RP(m+l+1)/RP(m))) \oplus \tilde{J}(L_0^m(n)) \longrightarrow \tilde{J}(D_n(m, l))$$

defined by

$$\theta(J(x), J(y), J(z)) = J(p^!(x) + q_1^! f^!(y) + \mu(z))$$

is an isomorphism.

(2) If $l \equiv 2 \pmod 4$, *then the map*

$$\theta: \widetilde{J}(RP(l)) \oplus \widetilde{J}(S^m \wedge (RP(m+l+1)/RP(m))) \oplus \widetilde{J}(L_0^m(n)) \\ \oplus \widetilde{J}(S^l \wedge L_0^m(n)) \longrightarrow \widetilde{J}(D_n(m,l))$$

defined by

$$\theta(J(x), J(y), J(z), J(w)) = J(p^!(x) + q^! f^!(y) + \mu(z) + \nu(w))$$

is an isomorphism.

Remark. The partial result for $m \equiv 3 \pmod 4$, $l \equiv 7 \pmod 8$ and odd prime n is obtained in [7, Theorem 2. 3]. The method used in the proof of [7] is available for the case $\widetilde{KO}(S^m \wedge (RP(m+l+1)/RP(m))) = 0$ and $l \not\equiv 2 \pmod 4$.

3. Determination of $\widetilde{J}(D_n(m, l))$ for odd prime n. In this section we shall determine the structure of the direct summands of $\widetilde{J}(D_p(m, l))$ given in Theorem 2. 4, where p is an odd prime.

The first direct summand $\widetilde{J}(RP(l))$ has been known in Adams [3]: $\widetilde{J}(RP(l))$ is a cyclic group of order $2^{\phi(l,0)}$ generated by $J(\lambda)$.

And the third direct summand $\widetilde{J}(L_0^m(p))$ has been known in Kambe, Matsunaga and Toda [11]: $\widetilde{J}(L_0^m(p))$ is a cyclic group of order $p^{[m/(p-1)]}$ generated by $J(\bar{\sigma})$.

In order to determine the second direct summand $\widetilde{J}(S^m \wedge (RP(m+l+1)/RP(m)))$ we recall first the following Propositions.

Proposition 3.1 ([1, Theorem 7. 4]). *If* $m \not\equiv 3 \pmod 4$, *then* $\widetilde{KO}(RP(l)/RP(m))$ *is a cyclic group of order* $2^{\phi(l,m)}$ *generated by* $\lambda^{(\phi(m,0)+1)}$ *which maps into* $\lambda^{\phi(m,0)+1} \in \widetilde{KO}(RP(l))$ *by the projection. Moreover the Adams operations are given by*

$$\psi^k \lambda^{(\phi(m,0)+1)} = \begin{cases} 0 & (k:\ \textit{even}) \\ \lambda^{(\phi(m,0)+1)} & (k:\ \textit{odd}). \end{cases}$$

Proposition 3.2 ([1, Corollary 5. 3]). *Let X be a finite CW-complex. Then the following diagrams*

$$\begin{array}{ccc} \widetilde{K}(X) & \xrightarrow{I_C} & \widetilde{K}(S^2 \wedge X) \\ \psi_C^k \downarrow & & \downarrow \psi_C^k \\ \widetilde{K}(X) & \xrightarrow{kI_C} & \widetilde{K}(S^2 \wedge X), \end{array} \qquad \begin{array}{ccc} \widetilde{KO}(X) & \xrightarrow{I_R} & \widetilde{KO}(S^8 \wedge X) \\ \psi^k \downarrow & & \downarrow \psi^k \\ \widetilde{KO}(X) & \xrightarrow{k^4 I_R} & \widetilde{KO}(S^8 \wedge X) \end{array}$$

are commutative, wher ψ_C^k (*resp.* ψ^k) *is the Adams operation and* I_C (*resp.* I_R) *is the Bott isomorphism in K-theory* (*resp. KO-theory*).

Let $\nu_q(m)$ denote the exponent of the prime q in the prime power decomposition of m. Then we have

Lemma 3.3. *Let q be a prime. Given a non-negative integer i, we put $g(i)$ to be the greatest common divisor of $\{(k+qj)^i - k^i \mid j, k \in Z, 0 < k < q\}$. Then we have*

$$\nu_q(g(i)) = \begin{cases} \nu_q(i) + 2 & (q = 2 \text{ and } i \equiv 0 \pmod 2) \\ \nu_q(i) + 1 & (\text{otherwise}). \end{cases}$$

Proof. Assume that $q = 2$, $i \equiv 0 \pmod 2$ and $w = 1 + 2j$. We have an equality $w^{2v} - 1 = (w-1)(w+1)((w^2)^{v-1} + (w^2)^{v-2} + \cdots + 1)$. Since $(w^2)^{v-1} + (w^2)^{v-2} + \cdots + 1$ is odd for each odd integer v, we see that the lemma is true for the case $\nu_2(i) = 1$. Let u be a positive integer and v an odd integer. Then $w^{2^{u+1}v} - 1 = (w^{2^u v} - 1)(w^{2^u v} + 1)$, where $w^{2^u v} + 1 \equiv 2 \pmod 4$. Thus we can proceed by the induction with respect to $\nu_2(i)$. Similarly we can prove the other cases. q. e. d.

Let $\mathfrak{m}(t)$ be the function defined on positive integers as follows (cf. [3, p. 139]):

$$\nu_q(\mathfrak{m}(t)) = \begin{cases} 0 & \text{if } q \neq 2,\ t \not\equiv 0 \pmod{(q-1)} \\ 1 + \nu_q(t) & \text{if } q \neq 2,\ t \equiv 0 \pmod{(q-1)} \\ 1 & \text{if } q = 2,\ t \not\equiv 0 \pmod 2 \\ 2 + \nu_2(t) & \text{if } q = 2,\ t \equiv 0 \pmod 2. \end{cases} \tag{3.4}$$

Then we obtain

Theorem 3.5. (1) *If $m \equiv 0 \pmod 4$, then*

$$\tilde{J}(S^m \wedge (RP(m+l+1)/RP(m))) \cong Z_{2^h},$$

where $h = \min\{\phi(2m+l+1, 2m),\ \nu_2(m) + 1\}$.

(2) *If $m \not\equiv 0 \pmod 4$, the groups $\tilde{J}(S^m \wedge (RP(m+l+1)/RP(m)))$ are tabled as follows, where $N(m, l) = \mathfrak{m}((2m+l+1)/2)$:*

l (mod 8) \ m (mod 4)	0	1	2	3	4	5	6	7
1	0	$(N(m,l))$	0	0	0	$(N(m,l))$	0	0
2	0	0	0	(2)	(2)⊕(2)	(2)	0	0
3	0	$(N(m,l))$	(2)	(2)⊕(2)	(2)	$(N(m,l))$	0	0

Proof. (1) Let $m\equiv 0 \pmod 8$. Then, by Proposition 3.1 $\widetilde{KO}(S^m\wedge(RP(m+l+1)/RP(m)))\cong Z_{2^{\phi(m+l+1,m)}}$ is generated by $I_R^{m/8}(\lambda^{(\phi(m,0)+1)})$. By Proposition 3.2 we have $\psi^k\circ I_R^{m/8}=k^{m/2}I_R^{m/8}\circ\psi^k$ $(k\in Z)$, and hence for $x\in\widetilde{KO}(S^m\wedge(RP(m+l+1)/RP(m)))$

$$\psi^k(x)=\begin{cases}0 & (k\text{: even})\\ k^{m/2}x & (k\text{: odd}).\end{cases}$$

Therefore we have

$$\begin{aligned}&\sum_k(\bigcap_e k^e(\psi^k-1)\widetilde{KO}(S^m\wedge(RP(m+l+1)/RP(m))))\\ &=\sum_{k:\text{odd}}(\psi^k-1)\widetilde{KO}(S^m\wedge(RP(m+l+1)/RP(m)))\\ &=\sum_{k:\text{odd}}(k^{m/2}-1)\widetilde{KO}(S^m\wedge(RP(m+l+1)/RP(m))).\end{aligned}$$

Now, using (2.1), (2.3) and Lemma 3.3, it follows that

$$\widetilde{J}(S^m\wedge(RP(m+l+1)/RP(m)))\cong Z_{2^h},$$

where $h=\min\{\phi(m+l+1,m),\ \nu_2(m)+1\}$.

Let $m\equiv 4 \pmod 8$. Then we have the following short exact sequence:

$$0\longrightarrow\widetilde{KO}(S^4\wedge(RP(m+l+1)/RP(m)))\longrightarrow\widetilde{KO}(S^4\wedge RP(m+l+1))\longrightarrow\widetilde{KO}(S^4\wedge RP(m))\longrightarrow 0.$$

Hence, using [1, Corollary 5.2] and [5, Theorem 1.2)], it follows that $\widetilde{KO}(S^4\wedge(RP(m+l+1)/RP(m)))\cong Z_{2^{\phi(2m+l+1,2m)}}$ and for $x\in\widetilde{KO}(S^4\wedge RP(m+l+1)/RP(m)))$

$$\psi^k(x)=\begin{cases}0 & (k\text{: even})\\ k^2x & (k\text{: odd}).\end{cases}$$

This implies that for $x\in\widetilde{KO}(S^m\wedge(RP(m+l+1)/RP(m)))$

$$\psi^k(x)=\begin{cases}0 & (k\text{: even})\\ k^{m/2}x & (k\text{: odd}).\end{cases}$$

The rest of the proof for this case is quite similar to that for the case $m\equiv 0 \pmod 8$.

(2) Inspect the following commutative diagram, in which the rows and columns are exact:

$$\begin{array}{ccc} & & \widetilde{KO}(S^m\wedge(RP(m+l-1)/RP(m)))\\ & & \uparrow\\ \widetilde{KO}(S^m\wedge RP(m+l+1)/RP(m+l))) & \xrightarrow{q_1} & \widetilde{KO}(S^m\wedge(RP(m+l+1)/RP(m)))\\ \| & & \uparrow\\ \widetilde{KO}(S^m\wedge RP(m+l+1)/RP(m+l))) & \xrightarrow{q_2} & \widetilde{KO}(S^m\wedge RP(m+l+1)/RP(m+l-1)))\end{array}$$

$$
\begin{array}{l}
= \widetilde{KO}(S^m \wedge (RP(m+l-1)/RP(m))) \\
\qquad\qquad \uparrow \\
\xrightarrow{i_1} \widetilde{KO}(S^m \wedge (RP(m+l)/RP(m))) \\
\qquad\qquad \uparrow \\
\xrightarrow{i_2} \widetilde{KO}(S^m \wedge RP(m+l)/RP(m+l-1))).
\end{array}
$$

First we assume that $m+l+1$ is odd. Then there exists a homotopy equivalence $g: RP(m+l+1)/RP(m+l-1) \longrightarrow S^{m+l} \vee S^{m+l+1}$, which makes the following diagram homotopy commutative:

$$
\begin{array}{ccccc}
RP(m+l+1)/RP(m+l) & \leftarrow & RP(m+l+1)/RP(m+l-1) & \leftarrow & RP(m+l)/RP(m+l-1) \\
\downarrow \approx & & \downarrow g & & \downarrow \approx \\
S^{m+l+1} & \xleftarrow{q_3} & S^{m+l} \vee S^{m+l+1} & \xleftarrow{i_3} & S^{m+l},
\end{array}
$$

where i_3 is the inclusion map and q_3 is defined by $q_3(x) = *$ for $x \in S^{m+l}$. Therefore, we have the split exact sequence

$$
0 \to \widetilde{KO}(S^{2m+l+1}) \xrightarrow{q_2} \widetilde{KO}(S^m \wedge RP(m+l+1)/RP(m+l-1))) \xrightarrow{i_2} \widetilde{KO}(S^{2m+l}) \to 0
$$

of ψ-maps.

Especially, in case $m \equiv 2 \pmod 4$ and $l \equiv 4 \pmod 8$, we have $\widetilde{KO}(S^m \wedge RP(m+l-1)/RP(m))) = 0$ and $\widetilde{KO}(S^m \wedge RP(m+l+1)/RP(m))) \cong Z_2 \oplus Z_2$ by Proposition 1. 5. Hence we obtain the split exact sequence

$$
\begin{array}{l}
0 \longrightarrow \widetilde{KO}(S^{2m+l+1}) \xrightarrow{q_1} \widetilde{KO}(S^m \wedge RP(m+l+1)/RP(m))) \\
\xrightarrow{i_1} \widetilde{KO}(S^m \wedge (RP(m+l)/RP(m))) \longrightarrow 0
\end{array}
$$

of ψ-maps. It follows from Lemma 2. 2 that $\widetilde{J}(S^m \wedge (RP(m+l+1)/RP(m))) \cong \widetilde{J}(S^{2m+l+1}) \oplus \widetilde{J}(S^m \wedge (RP(m+l)/RP(m)))$. Moreover, the Adams operations on $\widetilde{KO}(S^m \wedge (RP(m+l)/RP(m)))$ are given by $\psi^k = k^{(2m+l)/2}$. And the fact $\widetilde{J}(S^m \wedge (RP(m+l)/RP(m))) \cong Z_2$ follows from Propositition 1. 5. This and the fact $\widetilde{J}(S^{2m+l+1}) \cong Z_2$ (cf. [3, p 146]) imply the part of $m \equiv 2 \pmod 4$ and $l \equiv 3,\ 4 \pmod 8$ in the table. Similarly, we can determine the case $m \equiv 3 \pmod 4$ and $l \equiv 2, 3 \pmod 8$.

When $m \equiv 3 \pmod 4$ and $l \equiv 5 \pmod 8$, we have the exact sequence

$$
\begin{array}{l}
Z \cong \widetilde{KO}(S^{2m+l+1}) \xrightarrow{q_1} \widetilde{KO}(S^m \wedge RP(m+l+1)/RP(m))) \\
\xrightarrow{i_1} \widetilde{KO}(S^m \wedge (RP(m+l)/RP(m))) \longrightarrow 0,
\end{array}
$$

where $\widetilde{KO}(S^m \wedge RP(m+l+1)/RP(m))) \cong Z$ and $\widetilde{KO}(S^m \wedge RP(m+l)/RP(m)))$ $\cong Z_2$ by Proposition 1. 5. Then, it is easy to see that the Adams operations on $\widetilde{KO}(S^m \wedge (RP(m+l+1)/RP(m)))$ and $\widetilde{KO}(S^m \wedge (RP(m+l)/RP(m)))$ are given by $\psi^k = k^{(2m+l+1)/2}$. This implies that $\widetilde{J}(S^m \wedge RP(m+l)/RP(m)))$ $\cong Z_2$ and $\widetilde{J}(S^m \wedge (RP(m+l+1)/RP(m))) \cong Z_{N(m,l)}$ by the same way as [3, p 147]. This shows the part of $m \equiv 3$ (mod 4) and $l \equiv 4, 5$ (mod 8) in the table.

The rest is similar to the above. q. e. d.

Finally, we determine the group $\widetilde{J}(S^{2l} \wedge L_0^m(p))$. To this end, we borrow the following from Kambe [10].

Propositon 3.6. (1) *$K(L_0^m(p))$ is a ring generated by σ with relations $(1+\sigma)^p = 1$ and $\sigma^{m+1} = 0$.*

(2) *$\widetilde{K}(L_0^m(p))$ is the direct sum of cyclic groups generated by σ, σ^2, $\cdots$, σ^{p-1}. Let $m = r(p-1)+s$, $0 \leqq s < p-1$. Then the order* of *σ^i is p^{r+1} or p^r according as $0 \leqq i < s$ or $s < i \leqq p-1$.*

In advance of proving our final theorem, we state the next lemma.

Lemma 3.7. *Let i and k be positive integers with $k \leqq i$. Then it holds*

$$\sum_{j=1}^{i} \binom{i}{j} (-1)^{i-j} j^k = \begin{cases} 0 & (k < i) \\ i! & (k = i). \end{cases}$$

Proof. For each k, consider $f_k(x) = \sum_{j=1}^{i} \binom{i}{j} (-1)^{i-j} j^k x^j$. Then $f_1(x) = ix(x-1)^{i-1}$ and $f_{k+1}(x) = x \dfrac{d}{dx} f_k(x)$. Therefore we can show that there exists $g_k(x) \in Z[x]$ such that $f_k(x) = g_k(x)(x-1)^{i-k+1} + (i!/(i-k)!)x^k(x-1)^{i-k}$ by the induction on k. Noting that $f_k(1) = \sum_{j=1}^{i} \binom{i}{j} (-1)^{i-j} j^k$, we readily see the lemma.

Theorem 3.8. *Let $l = t(p-1)+w$ for $0 \leqq w < p-1$. Then $\widetilde{J}(S^{2l} \wedge L_0^m(p))$ is a cyclic group of order p^h generated by $J \circ r(I_C^l(\sigma))$, where $h = min\{\nu_p(l)+1, [(m+w)/(p-1)]\}$.*

Proof. Consider the real restriction r and the J-homomorphism

$$\widetilde{K}(S^{2l} \wedge L_0^m(p)) \xrightarrow{r} \widetilde{KO}(S^{2l} \wedge L_0^m(p)) \xrightarrow{J} \widetilde{J}(S^{2l} \wedge L_0^m(p)).$$

Since $\widetilde{KO}(S^{2l} \wedge L_0^m(p))$ is of odd order, r and $J \circ r$ are epimorphic.

Moreover, the Adams operations commute with the real restriction [4, Lemma A 2]. Therefore $\ker J\circ r$ is generated by the elements of $\ker r$ and $\sum_k(\bigcap_e k^e(\psi_C^k-1)\widetilde{K}(S^{2l}\wedge L_0^m(p)))$:

$$\ker J\circ r = \langle \ker r \cup \sum_k(\bigcap_e k^e(\psi_C^k-1)\widetilde{K}(S^{2l}\wedge L_0^m(p)))\rangle .$$

Put $x_i = I_C^l(\eta^i - 1)\in \widetilde{K}(S^{2l}\wedge L_0^m(p))$. Then it follows from Proposition 3.6 that

$$x_{i+p} = x_i \tag{3.9}$$

and

$$\widetilde{K}(S^{2l}\wedge L_0^m(p)) = \langle \{x_i\mid 0<i<p\}\rangle .$$

By Proposition 3.2 we have

$$\psi_C^k(x_i) = k^l x_{ki}. \tag{3.10}$$

Let $c\colon KO\longrightarrow K$ and $t\colon K\longrightarrow K$ be the complexification and conjugation. Then $t+1=c\circ r$ and $r=r\circ t$. Hence $r((1-t)x)=0$ for $x\in\widetilde{K}(S^{2l}\wedge L_0^m(p))$. Conversely, assume $r(y)=0$. Then $y+t(y)=c\circ r(y)=0$. Since $\widetilde{K}(S^{2l}\wedge L_0^m(p))$ is of odd order, $y=2x$ for some $x\in\widetilde{K}(S^{2l}\wedge L_0^m(p))$, and the equality $2y=y-t(y)=2(1-t)x$ implies $y=(1-t)x$. Therefore $\ker r=(1-t)\widetilde{K}(S^{2l}\wedge L_0^m(p))$. Since $t(x_i)=\psi_C^{-1}(x_i)=(-1)^l x_{p-i}$ by (3.9) and (3.10), we have

$$\ker r = \langle \{(-1)^l x_{p-i} - x_i\mid 0<i<p\}\rangle. \tag{3.11}$$

$\widetilde{K}(S^{2l}\wedge L_0^m(p))$ is of order p^m. This implies that $\bigcap_e k^e(\psi_C^k-1)\widetilde{K}(S^{2l}\wedge L_0^m(p))$ is 0 or $(\psi_C^k-1)\widetilde{K}(S^{2l}\wedge L_0^m(p))$ according as $k\equiv 0 \pmod p$ or $k\not\equiv 0 \pmod p$. And $(\psi_C^k-1)\widetilde{K}(S^{2l}\wedge L_0^m(p)) = \langle\{k^l x_{ki}-x_i\mid 0<i<p\}\rangle$ by (3.10). Thus $\sum_k(\bigcap_e k^e(\psi_C^k-1)\widetilde{K}(S^{2l}\wedge L_0^m(p))$ is generated by $A_1=\{k^l x_{ki}-x_i\mid 0<i<p,\ k\not\equiv 0 \pmod p\}$. Since A_1 contains the generators of $\ker r$ in (3.11), we have

$$\ker J\circ r = \langle A_1\rangle .$$

Choose an integer N_k with $N_k k^l\equiv 1 \pmod{p^m}$ for each $k\not\equiv 0 \pmod p$. Then we have $N_k(k^l x_k - x_1) = x_k - N_k x_1$, and $(N_k - N_{k+pj})x_1 = (x_{k+pj} - N_{k+pj}x_1) - (x_k - N_k x_1)$ by (3.9). Thus, $\ker J\circ r$ contains $A_2=\{x_k - N_k x_1\mid 0<k<p\}$ and $A_3=\{(N_k - N_{k+pj})x_1\mid 0<k<p,\ j\in Z\}$. Conversely, every element in A_1 is a linear combination of the elements in $A_2\cup A_3$. Hence $\ker J\circ r=\langle A_2\cup A_3\rangle$. Thus

$$\widetilde{J}(S^{2l}\wedge L_0^m(p)) = \langle\{J\circ r(x_1)\}\rangle . \tag{3.12}$$

To determine the order of $J\circ r(x_1)$ we set $y_i = I_C^i(\sigma^i)\in\widetilde{K}(S^{2l}\wedge L_0^m(p))$. Then $\widetilde{K}(S^{2l}\wedge L_0^m(p))$ is the direct sum of cyclic groups generated by $y_1=x_1, y_2, \cdots, y_{p-1}$, and the order of y_i is p^{r+1} or p^r according as $0<i\leqq s$ or $s<i<p$, where r and s are those of Proposition 3.6 (2). By the equality $(\eta-1)^i=\sum_{j=1}^{i}\binom{i}{j}(-1)^{i-j}(\eta^j-1)$, we have $y_i=\sum_{j=1}^{i}\binom{i}{j}(-1)^{i-j}x_j=\sum_{j=1}^{i}\binom{i}{j}(-1)^{i-j}(x_j-N_jx_1)+(\sum_{j=1}^{i}\binom{i}{j}(-1)^{i-j}N_j)x_1$. Therefore $\langle A_2\rangle$ coincides with the subgroup generated by $A_4=\{y_i-(\sum_{j=1}^{i}\binom{i}{j}(-1)^{i-j}N_j)x_1\,|\,0<i<p\}$. This together with the above remark on the order of y_i enables us to see that $\langle\{x_1\}\rangle\cap\langle A_4\rangle$ coincides with the subgroup generated by
$A_5=\{p^{r+1}(\sum_{j=1}^{i}\binom{i}{j}(-1)^{i-j}N_j)x_1\,|\,0<i\leqq s\}\cup\{p^r(\sum_{j=1}^{i}\binom{i}{j}(-1)^{i-j}N_j)x_1\,|\,s<i<p\}$.

Denote by H the quotient group of $\widetilde{K}(S^{2l}\wedge L_0^m(p))$ by $\langle A_2\rangle$. Then ord H is p^{r+1} if $\sum_{j=1}^{i}\binom{i}{j}(-1)^{i-j}N_j\equiv 0 \pmod p$ for $s<i<p$ and p^r if $\sum_{j=1}^{i}\binom{i}{j}(-1)^{i-j}N_j\not\equiv 0 \pmod p$ for some i with $s<i<p$.

If $j\not\equiv 0 \pmod p$, we have $j^{p-1}\equiv 1 \pmod p$, and hence $j^l\equiv j^w \pmod p$. Therefore, by the definition of N_j, we have $N_j\equiv j^{p-1-w} \pmod p$. Thus, by making use of Lemma 3.7, we see that

$$\text{ord } H = p^{[(m+w)/(p-1)]}.$$

The greatest common divisor of $p^{[(m+w)/(p-1)]}$ and the integers N_k-N_{k+pj} $(0<k<p,\ j\in Z)$ equals $p^{\min\{\nu_p(l)+1,[(m+w)/(p-1)]\}}$ by Lemma 3.3, because we have $k^l(k+pj)^l(N_k-N_{k+pj})\equiv(k+pj)^l-k^l \pmod{p^m}$ for $0<k<p$. Thus the order of $J\circ r(x_1)$ equals $p^{\min\{\nu_p(l)+1,[(m+w)/(p-1)]\}}$. q. e. d.

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