Math. J. Okayama Univ. 22 (1980), 195—204

ON CONFORMAL KILLING FORMS AND THE PROPER SPACE OF Δ FOR p-FORMS

MITSURU KORA

In 1975, S. Gallot and D. Meyer [2] discussed the proper value λ of Δ for p-forms in compact Riemannian manifolds of positive curvature operator and found its lower bound λ_1. Recently, S. Tachibana and S. Yamaguchi [10] have investigated and determined the proper space V_{λ_1} of such a manifold in terms of Killing, closed conformal Killing and special Killing p-forms. On the other hand, H. Maillot [5] studied the dual $^*\omega$ of a Killing p-form ω. In connection with this, H. Yanamoto [11] has conjectured that in any orientable Riemannian manifold M^n the dual $^*\omega$ of a conformal Killing p-form ω would be conformal Killing, and proved it when $n=3$ and $p=1$.

In this paper, we show the conjecture to be true and express the space V_{λ_1} in a better form by making use of this duality.

The author wishes to express his gratitude to Prof. S. Tachibana who gave him useful comments and continuous encouragement.

1. Preliminaries. Let M^n $(n>1)$ be an n dimensional Riemannian manifold. Denote by $g=(g_{ab})$, $R_{abc}{}^e$ and R_{ab} respectively the metric, the curvature and the Ricci tensor, where the indices $a, b, c, \cdots$ run over the range $\{1, 2, \cdots, n\}$. We represent tensors by their components with respect to the natural base and the summation convention is assumed throughout the paper.

A skew symmetric covariant tensor $\omega=(\omega_{a_1\cdots a_p})$ is identified with the differential p-form

$$\omega=\frac{1}{p!}\,\omega_{a_1\cdots a_p}dx^{a_1}\wedge\cdots\wedge dx^{a_p},$$

and the coefficients of its exterior differential $d\omega$ and the co-differential $\delta\omega$ are given dy

$$(d\omega)_{a_1\cdots a_{p+1}}=\sum_{i=1}^{p+1}(-1)^{i+1}\nabla_{a_i}\omega_{a_1\cdots\hat{a}_i\cdots a_{p+1}}$$

$$(\delta\omega)_{a_2\cdots a_p}=-\nabla^b\omega_{ba_2\cdots a_p}$$

where $\nabla^b=g^{ba}\nabla_a$, ∇_a denotes the operator of covariant differentiation and $\hat{a}_i$ means a_i to be deleted.

By the natural identification with respect to g, a vector field Y on M^n

can be regarded as a 1-form which is denoted by Y again. $e(Y)\omega$ and $i(Y)\omega$ denote respectively the exterior and the interior product of ω by Y and they have the components

$$(e(Y)\omega)_{a_1\cdots a_{p+1}} = \sum_{i=1}^{p+1} (-1)^{i-1} Y_{a_i} \omega_{a_1\cdots\hat{a}_i\cdots a_{p+1}}$$

$$(i(Y)\omega)_{a_2\cdots a_p} = Y^b \omega_{ba_2\cdots a_p}.$$

Denoting by $\Delta = d\delta + \delta d$ the Laplacian operator, we have

$$(1.1) \qquad (\Delta\omega)_{a_1\cdots a_p} = -\nabla^b\nabla_b\omega_{a_1\cdots a_p} + H(\omega)_{a_1\cdots a_p}$$

as the coefficients of $\Delta\omega$, where $H(\omega)_{a_1\cdots a_p}$ are the components of $H(\omega)$ given by $H(\omega)_a = R_{ab}\omega^b$ for $p = 1$ and

$$(1.2) \qquad H(\omega)_{a_1\cdots a_p} = \sum_{i=1}^{p} R_{a_i}{}^b \omega_{a_1\cdots \underset{(i)}{b} \cdots a_p} + \sum_{i<j} R_{a_i a_j}{}^{bc} \omega_{a_1\cdots \underset{(i)}{b} \cdots \underset{(j)}{c} \cdots a_p}$$

for $n \geq p \geq 2$. (1. 1) may de written as

$$(1.3) \qquad \Delta\omega = -\nabla^b\nabla_b\omega + H(\omega).$$

The quadratic form $F_p(\omega)$ of ω is defined by

$$F_p(\omega) = \langle H(\omega), \omega\rangle$$

$$= \frac{1}{(p-1)!}\left(R_{ab}\omega^a{}_{a_2\cdots a_p}\omega^{ba_2\cdots a_p} + \frac{p-1}{2} R_{abce}\omega^{ab}{}_{a_3\cdots a_p}\omega^{cea_3\cdots a_p}\right)$$

and it appears in the well known formula which is valid for any p-form ω:

$$(1.4) \qquad F_p(\omega) = \langle \Delta\omega, \omega\rangle - |\nabla\omega|^2 - \frac{1}{2}\Delta|\omega|^2,$$

where we have put for p-forms ω and τ, $|\omega|^2 = \langle\omega, \omega\rangle$,

$$\langle\omega, \tau\rangle = \frac{1}{p!}\omega_{a_1\cdots a_p}\tau^{a_1\cdots a_p},$$

$$|\nabla\omega|^2 = \frac{1}{p!}\nabla_b\omega_{a_1\cdots a_p}\nabla^b\omega^{a_1\cdots a_p}.$$

If a non-zero p-form ω satisfies $\nabla\omega = \lambda\omega$ with a constant λ, it is called a proper form of Δ corresponding to the proper value λ. The space of all proper p-forms corresponding to λ is denoted by V^p_λ. If M^n is compact and orientable, the decomposition $V^p_\lambda = (V^p_\lambda \cap d^{-1}(0)) \oplus (V^p_\lambda \cap \delta^{-1}(0))$ holds for $\lambda \neq 0$ from the decomposition theorem of Hodge-de Rham.

2. The Killing and the conformal Killing p-forms. Following [7] and [4], a p-form ω is called conformal Killing, if there exists a $(p-1)$-form ρ satisfying

$$\begin{aligned} &\nabla_b \omega_{a_1\cdots a_p} + \nabla_{a_1}\omega_{b a_2 \cdots a_p} \\ &= 2\rho_{a_2\cdots a_p} g_{b a_1} - \sum_{i=2}^{p} (-1)^i (\rho_{a_1\cdots \hat{a}_i \cdots a_p} g_{b a_i} + \rho_{b a_2 \cdots \hat{a}_i \cdots a_p} g_{a_i a_1}). \end{aligned} \tag{2.1}$$

As was shown in [4], ω and ρ satisfy

$$\rho = -\frac{1}{n-p+1}\delta\omega \tag{2.2}$$

$$\nabla_b \omega_{a_1\cdots a_p} = \frac{1}{p+1}(d\omega)_{b a_1 \cdots a_p} - \frac{1}{n-p+1}\sum_{i=1}^{p}(-1)^{i-1} g_{a_i b}(\delta\omega)_{a_1\cdots \hat{a}_i \cdots a_p} \tag{2.3}$$

$$\begin{aligned} &\nabla_b \nabla_c \omega_{a_1\cdots a_p} + U(\omega)_{cb, a_1\cdots a_p} \\ &= \nabla_b \rho_{a_2\cdots a_p} g_{a_1 c} + \nabla_c \rho_{a_2 \cdots a_p} g_{a_1 b} - \nabla_{a_1}\rho_{a_2\cdots a_p} g_{bc} \\ &\quad - \frac{1}{2}\sum_{i=2}^{p}(-1)^i (\nabla_b \rho_{c a_2 \cdots \hat{a}_i \cdots a_p} + \nabla_c \rho_{b a_2 \cdots \hat{a}_i \cdots a_p}) g_{a_1 a_i} \\ &\quad - \frac{1}{2}\sum_{i=2}^{p}(-1)^i (\nabla_b \rho_{a_1 \cdots \hat{a}_i \cdots a_p} - \nabla_{a_1}\rho_{b a_2 \cdots \hat{a}_i \cdots a_p}) g_{c a_i} \\ &\quad - \frac{1}{2}\sum_{i=2}^{p}(-1)^i (\nabla_c \rho_{a_1 \cdots \hat{a}_i \cdots a_p} - \nabla_{a_1}\rho_{c a_2 \cdots \hat{a}_i \cdots a_p}) g_{b a_i}, \end{aligned} \tag{2.4}$$

where we have put

$$\begin{aligned} U(\omega)_{bc, a_1\cdots a_p} &= \frac{1}{2}\sum_{i=1}^{p} R_{bca_i}{}^{e}\omega_{a_1 \cdots \underset{(i)}{e} \cdots a_p} - \frac{1}{2}(R_{ba_1c}{}^{e} + R_{ca_1b}{}^{e})\omega_{e a_2 \cdots a_p} \\ &\quad - \frac{1}{2}\sum_{i=2}^{p}(R_{ba_1a_i}{}^{e}\omega_{c a_2 \cdots \underset{(i)}{e} \cdots a_p} + R_{ca_1a_i}{}^{e}\omega_{b a_2 \cdots \underset{(i)}{e} \cdots a_p}). \end{aligned} \tag{2.5}$$

If we transvect (2. 4) with g^{bc}, we get

$$\begin{aligned} &\nabla^b \nabla_b \omega_{a_1\cdots a_p} + U(\omega)^b{}_{b, a_1\cdots a_p} \\ &= -(n-p)\nabla_{a_1}\rho_{a_2\cdots a_p} + (d\rho)_{a_1\cdots a_p}. \end{aligned} \tag{2.6}$$

Taking the skew symmetric part in (2. 6) with respect to the indices $a_1, \cdots, a_{p-1}$ and a_p and substituting (1. 2) and (2. 2), we have

$$p\nabla^b \nabla_b \omega + H(\omega) + \frac{2p-n}{n-p+1} d\delta\,\omega = 0. \tag{2.7}$$

A p-form ω is called Killing [9], if $\nabla\omega$ is a $(p+1)$-form. It follows from (2. 1), (2. 2) and (2. 3) that ω is Killing if and only if it is co-closed conformal Killing, and consequently, any Killing p-form ω satisfies

$$\nabla_b \nabla_c \omega_{a_1 \cdots a_p} + U(\omega)_{bc, a_1 \cdots a_p} = 0 . \tag{2. 8}$$

We are interested in the vector spaces with natural structure defined by

C^p = the space of all conformal Killing p-forms,

$C^p(d) = C^p \cap d^{-1}(0)$ = the space of all closed conformal Killing p-forms

$K^p = C^p \cap \delta^{-1}(0)$ = the space of all *K*illing p-forms.

Lemma 1 (cf. [10, Lemma 2. 5] and [5, §1]). *Let ω be a p-form, and Y any vector field. We then have the following*:

(i) $\omega \in C^p$ *if and only if* ω *satifies*

$$\nabla_Y \omega = \frac{1}{p+1} i(Y) d\omega - \frac{1}{n-p+1} e(Y) \delta\omega . \tag{2. 9}$$

(ii) $\omega \in K^p$ *if and only if* ω *satisfies*

$$\nabla_Y \omega = \frac{1}{p+1} i(Y) d\omega. \tag{2. 10}$$

(iii) $\omega \in C^p(d)$ *if and only if* ω *satisfies*

$$\nabla_Y \omega = - \frac{1}{n-p+1} e(Y) \delta\omega . \tag{2. 11}$$

Proof. (i) If ω satisfies (2. 3), by interchanging the indices b and a_1 in (2. 3) and taking their sum, we have (2. 1) with the $(p-1)$-form $-(n-p+1)\rho = \delta\omega$. This means that $\omega \in C^p$ if and only if ω satisfies (2. 3), or equivalently (2. 9).

(ii), (iii) (2. 10) and (2. 11) give us $\delta\omega = 0$ and $d\omega = 0$, respectively. The desired results then follow from (i).

Now we suppose that M^n is orientable. If we denote by W^* the dual space of a vector space W of p-forms with respect to *, we have the following.

Theorem 1. *In any orientable Riemannian manifold M^n,*

(i) $C^p = (C^{n-p})^*$

(ii) $C^p(d) = (K^{n-p})^*$ *or* $K^p = (C^{n-p}(d))^*$

hold for $n > p \geq 1$.

Proof. Since the volume element of M^n is parallel, we get $*\nabla_Y\omega = \nabla_Y{*\omega}$ for any p-form ω and any vector field Y. We also have $*i(Y)d\omega = -e(Y)\delta{*\omega}$ and $*e(Y)\delta\omega = -i(Y)d{*\omega}$. By means of these identities and Lemma 1, we obtain (i) and (ii).

Next we give an application of Theorem 1. As is well known, $\langle\Delta\omega, \omega\rangle = \langle\Delta{*\omega}, {*\omega}\rangle$, $|\omega|^2 = |{*\omega}|^2$ and $|\nabla\omega|^2 = |\nabla{*\omega}|^2$. Substituting them into (1. 4), we get

$$F_p(\omega) = F_{n-p}(*\omega) \tag{2. 12}$$

for any p-form ω ($n > p \geq 1$). If we take account of Theorem 1 and (2. 12), we can remove the assumption $p \geq \frac{n}{2}$ in Theorem 4 of Kashiwada's paper [4], i. e., we have

Corollary 1. *In a compact orientable Riemannian manifold* M^n, *if a conformal Killing p-form* ω $(n > p \geq 1)$ *satisfies* $F_p(\omega) \leq 0$, *then it is parallel. Especially, if* $F_p(\omega)$ *is negative definite, then there exists no conformal Killing p-form other than the zero form.*

Remark. Denote by $\theta(X) = di(X) + i(X)d$ the operator of Lie derivative by X. As is known in [3, p. 109], for a Killing vector field X, $\theta(X) = -\delta e(X) - e(X)\delta$ holds. Making use of this and Lemma 1, we can see that if $\omega \in C^p$ and X is a Killing vector field $\theta(X)\omega \in C^p$ holds.

3. The special Killing and the special conformal Killing *p*-forms. A p-form ω is called special with k, if it satisfies

$$U(\omega)_{bc, a_1 \cdots a_p} = k\Big(g_{bc}\omega_{a_1 \cdots a_p} - \sum_{i=1}^{p} g_{ba_i}\omega_{a_1 \cdots c \cdots a_p}\Big), \tag{3. 1}$$

with a constant k. The space of all the p-forms special with k is denoted by S_k^p. For example, any p-form in the space of constant sectional curveture k is special with k. The vector field which is dual to $\omega \in S_k^1$ with respect to g belongs to the k-nullity distribution of M^n.

$\omega \in S_k^p$ satisfies

$$H(\omega) = p(n-p)k\omega. \tag{3. 2}$$

In fact, if we transvect (2. 5) and (3. 1) with g^{bc} and take the skew symmetric parts with respect to the indices $a_1, \cdots, a_{p-1}$ and a_p, we get (3. 2), taking account of (1. 2).

By means of (3. 2) and (1. 3), we have for $\omega \in S_k^p$

$$\Delta\omega = -\nabla^b\nabla_b\omega + p(n-p)k\omega \,. \tag{3. 3}$$

Now we put

$C_k^p = C^p \cap S_k^p,\ K_k^p = K^p \cap S_k^p,\ C_k^p(d) = C^p(d) \cap S_k^p.$

Evidently, $K_k^p = C_k^p \cap \delta^{-1}(0)$ and $C_k^p(d) = C_k^p \cap d^{-1}(0)$ hold.

By (2. 6) and (3. 1), $\omega \in C_k^p$ satisfies

$$\begin{aligned} &\nabla^b\nabla_b\omega_{a_1\cdots a_p} + (n-p)k\omega_{a_1\cdots a_p} \\ &\qquad = -(n-p)\nabla_{a_1}\rho_{a_2\cdots a_p} + (d\rho)_{a_1\cdots a_p}\,. \end{aligned} \tag{3. 4}$$

Lemma 2. *$\omega \in K_k^p$ if and only if $\omega \in K^p$ and it satifies*

$$\nabla_Y d\omega = -k(p+1)e(Y)\omega \tag{3. 5}$$

for any vector field Y.

Proof. Let $\omega \in K^p$. By means of (2. 8) and (3. 1), $\omega \in S_k^p$ is equivalent to the establishment of the equation

$$\nabla_Z\nabla_Y\omega - \nabla_{\nabla_Z Y}\omega = -ki(Y)e(Z)\omega, \tag{3. 6}$$

for any vector fields Y and Z. On the other hand, (2. 10) and (3. 6) hold if and only if (2. 10) and (3. 5) hold.

Lemma 3 (cf. [10]). *In any orientable Riemannian manifold M^n,*

(i) $K_k^p = K^p \cap V^p_{(p+1)(n-p)k} \cap d^{-1}(C^{p+1}(d)) \quad (n > p \geq 1)$

(ii) $(k_k^{n-p})^* = C^p(d) \cap V^p_{p(n-p+1)k} \cap \delta^{-1}(k^{p-1}) \quad (n > p > 1)$

hold for any constant k.

Proof. Let $\omega \in C_k^p$. Making use of (2. 7), (3. 2) and (3. 3), we get

$$\Delta\omega = (p+1)(n-p)k\omega + \frac{2p-n}{p(n-p+1)}d\delta\omega\,. \tag{3. 7}$$

By virtue of $K_k^p = C_k^p \cap \delta^{-1}(0)$ and $C_k^p(d) = C_k^p \cap d^{-1}(0)$, we then have

$$K_k^p \subset V^p_{(p+1)(n-p)k} \text{ and } C_k^p(d) \subset V^p_{p(n-p+1)k}\,. \tag{3. 8}$$

(i) Let $\omega \in K_k^p$. Since $\omega \in V^p_{(p+1)(n-p)k}$ from (3. 8), $\delta d\omega = (p+1)(n-p)k\omega$ holds. Substituting this into (3. 5) and taking account of Lemma 1 (iii), we have $d\omega \in C^{p+1}(d)$. From this and (3. 8), we see the left hand of (i) is included in the righthand side. Conversely, if ω belongs to the right hand side of (i),

$$\nabla_Y d\omega = -\frac{1}{n-p}e(Y)\delta d\omega = -\frac{1}{n-p}e(Y)\Delta\omega = -k(p+1)e(Y)\omega$$

holds. Thus $\omega \in K_k^p$ by virtue of Lemma 2.

(ii) If we operate * to (i), change p to $n-p$, and take account of Theorem 1 (ii), we have (ii).

Lemma 4. *In any orientable Riemannian manifold* M^n,

(i) $C_k^p(d) = (K_k^{n-p})^*$ or $K_k^p = (C_k^{n-p}(d))^*$ $(n > p > 1)$

holds for any constant k. *Consequently, we have*

(ii) $C_k^p(d) = C^p(d) \cap V^p_{p(n-p+1)k} \cap \delta^{-1}(K^{p-1})$ $(n > p > 1)$.

Proof. Let $\omega \in C_k^p$. Interchanging the indices a_1 and a_2 in (3.4) and taking the sum, we get $\rho \in K^{p-1}$. By (2.2) this gives us $C_k^p \subset \delta^{-1}(K^{p-1})$ for $n \geq p > 1$. By virtue of this, (3.8) and Lemma 3 (ii), we see $C_k^p(d) \subset (K_k^{n-p})^*$.

Next we show the conversed inclusion $C_k^p(d) \supset (K_k^{n-p})^*$. Let $\omega \in (K_k^{n-p})^*$. Since $\omega \in C^p(d)$ from Lemma 3 (ii), it is sufficient to show $\omega \in S_k^p$. $*\omega \in K_k^{n-p}$ satisfies the equations corresponding to (3.5) and (3.6). Operating * to them, we thus have

$$\nabla_Y \delta\omega = k(n-p+1)\, i(Y)\omega, \tag{3.9}$$

$$\nabla_Z \nabla_Y \omega - \nabla_{\nabla_Z Y}\omega = -ke(Y)\, i(Z)\omega, \tag{3.10}$$

for every vector fields Y and Z. If we take account of (2.2) and substitute (3.9) and (3.10) into (2.4), we get (3.1).

From Lemma 4 (i) and Lemma 2, we have the following

Lemma 5. *In any orientable Riemannian manifold* M^n, $\omega \in C_k^p(d)$ *if and onlyif* $\omega \in C^p(d)$ *and it satisfies* (3.9) *for any vector field* Y.

By Lemma 4 (ii), $\omega \in C_k^p(d)$ satisfies $\delta\omega \in K^{p-1}$ and

$$\nabla_Y d\delta\omega = \nabla_Y \Delta\omega = kp(n-p+1)\nabla_Y\omega = -kpe(Y)\delta\omega$$

for any vector field Y. This means $C_k^p(d) \subset \delta^{-1}(K_k^{p-1})$ from Lemma 2. Similarly, by making use of Lemma 3 (i) and Lemma 5, we have $K_k^p \subset d^{-1}(C_k^{p+1}(d))$.

Lemma 6. *In any orientable Riemannian manifold* M^n, $C_k^p(d) \subset$

$\delta^{-1}(K_k^{p-1})$ *and* $K_k^p \subset d^{-1}(C_k^{p+1}(d))$ $(n > p > 1)$ *hold for any constant* k.

4. Compact orientable Riemannian manifold of postive curvature operator. In this section, we assume that M^n is of positive curvature operator, i. e., there exists a positive constant k such that $-R_{abce}\omega^{ab}\omega^{ce} \geq 2k\omega_{ab}\omega^{ab}$ holds for any 2-form ω. For example, Riemannian manifolds of constant curvature $k > 0$ are of positive curvature operator. For the study of M^n, we refer to [1, 2, 6 and 8]. Moreover if M^n is compact and orientable, it is known in [2] that the proper value λ of Δ for p-forms ω $(n \geq p \geq 1)$ satisfies

$$(4.1) \qquad \begin{aligned} &\lambda \geq p(n-p+1)k \overset{\text{put}}{=} \lambda_{d,p} \qquad \text{if} \quad d\omega = 0 \\ &\lambda \geq (p+1)(n-p)k \overset{\text{put}}{=} \lambda_{\delta,p} \qquad \text{if} \quad \delta\omega = 0. \end{aligned}$$

S. Tachibana and S. Yamaguchi [10] obtained that

$$(4.2) \qquad \begin{aligned} &V_{\lambda_{d,p}}^p \cap d^{-1}(0) = C^p(d) \cap \delta^{-1}(K_k^{p-1}) \qquad (n > p > 1). \\ &V_{\lambda_{\delta,p}}^p \cup \delta^{-1}(0) = K_k^p \qquad (n > p \geq 1). \end{aligned}$$

Theorem 2. *In a compact orientable Riemannian manifold* M^n *of positive curvature operator, we have*

$$V_{\lambda_{d,p}}^p \cap d^{-1}(0) = C_k^p(d) \qquad (n > p > 1).$$

Consequently, we have

$$V_{\lambda_{d,p}}^p = C_k^p(d) \qquad (n > 2p > 2).$$

Proof. Operating * to (4.2), changed p to $n-p$, and taking account of Lemma 4 (i), we have the former. To show the latter, we notice $V_{\lambda_{d,p}}^p = (V_{\lambda_{d,p}}^p \cap d^{-1}(0)) \oplus (V_{\lambda_{d,p}}^p \cap \delta^{-1}(0))$. On the other hand, if $n > 2p$, $d(V_{\lambda_{d,p}}^p \cap \delta^{-1}(0)) \subset V_{\lambda_{d,p}}^{p+1} \cap d^{-1}(0) = \{0\}$ holds by virtue of $\lambda_{d,p} < \lambda_{d,p+1}$ and (4.1). We thus have $V_{\lambda_{d,p}}^p \cap \delta^{-1}(0) = \{0\}$ for $n > 2p$.

Corollary 2. *In a compact orientable Riemannian manifold* M^n *of positive curvature operator, we have*

(i) $$C_k^p = K_k^p \oplus C_k^p(d) \qquad (n > p \geq 1).$$

Consequently, we have

(ii) $$C_k^p = (C_k^{n-p})^* \qquad (n > p \geq 1).$$

Proof. As was shown in the proof of Lemma 3, $\omega \in C^p_k$ satisfies (3. 7), that is,

$$\omega = \frac{1}{(p+1)(n-p)k}\,\delta d\omega + \frac{1}{p(n-p+1)k}\,d\delta\omega\,. \tag{4. 3}$$

It is not difficult to see that the closed part belongs to $V^p_{\lambda_{d,p}} \cap d^{-1}(0)$ and the co-closed part belongs to $V^p_{\lambda_{\delta,p}} \cap \delta^{-1}(0)$. We then have $C^p_k \subset K^p_k \oplus C^p_k(d)$ by virtue of (4. 2) and Theorem 2. The converse is obvious. (ii) follows from (i) and Lemma 4 (i).

If $n = 2m$, $\lambda_{d,m} = m(m+1)k = \lambda_{\delta,m}$ holds. From (4. 2), Theorem 1 and Corollary 2, we get

Corollary 3. *In a compact orientable Riemannian manifold* $M^{2m}(m>1)$ *of positive curvature operator, there holds*

$$V^m_{m(m+1)k} = C^m_k = K^m_k \oplus C^m_k(d)\,.$$

Remark. We don't know whether $\lambda_{d,p}$ and $\lambda_{\delta,p}$ are the first proper values of M^n or not. In other words, the spaces $V^p_{\lambda_{d,p}} \cap d^{-1}(0)$ and $V^p_{\lambda_{\delta,p}} \cap \delta^{-1}(0)$ may be $\{0\}$ for some M^n. But we see that if M^n is Euclidean n-sphere of curvature k then they are the first proper values, since $C^p(d) = C^p_k(d) \neq \{0\}$ and $K^p_k = K^p \neq \{0\}$.

REFERENCES

[1] M. BERGER: Sur les variétés à operateur de courbure positif, C. R. Acad. Sc. Paris **253** (1961), 2832—2834.

[2] S. GALLOT et D. MEYER: Opérateur de courbure et laplacien des formes differentielles d'une variété riemanniennes, J. Math. Pures Appl. **54** (1975), 259—284.

[3] S. I. GOLDBERG: Curvature and Homology, Academic Press, 1962.

[4] T. KASHIWADA: On conformal Killingtensor, Nat. Sci Rep. Ochanomizu Univ. **19** (1968), 67—74.

[5] H. MAILLOT: Sur les formes de Killing et leurs ≪dual≫ sur une variété riemannienne, C. R. Acad. Sc. Paris **276** (1973), 1565—1568.

[6] D. MEYER: Sur les variété riemanniennes à opérateur de coubure positif, C. R. Acad. Sc. Paris **272** (1971), 482—485.

[7] S. TACHIBANA: On conformal Killing tensor in a Riemannian space, Tôhoku Math. J. **21**(1969), 56—64.

[8] S. TACHIBANA: A theorem on Riemannian manifolds of positive curvature operator, Proc. Japan Acad. **50** (1974), 301—302.

[9] S. TACHIBANA and T. KASHIWADA: On the integrability of Killing-Yano's equation, J. Math. Soc. Japan **21** (1969), 259—265.

[10] S. TACHIBANA and S. YAMAGUCHI: On the first proper space of Δ for p-forms in compact Riemannian manifold of positive curvature operator, to appear.

[11] H. YANAMOTO: On conformal Killing tensors of degree 2 in 3-dimensional space. Res. Rep. Nagaoka Tech. Coll. **9** (1974), 191—196.

DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVEYSITY

(*Received February 28, 1980*)