Math. J. Okayama Univ. 22 (1980), 169—184

CARMICHAEL NUMBERS WITH MANY PRIME FACTORS

MASATAKA YORINAGA

The Carmichael number N is defined as a composite number $N>0$ such that for any positive integer a relatively prime to N, the congruence

$$a^{N-1}\equiv 1 \pmod N$$

is always true. These Carmichael numbers are characterized by the following

Theorem 1 (J. Chernick [2]). *An integer N is a Carmichael number if and only if N may be expressed as a product of distinct odd prime numbers $p_1, p_2, \cdots, p_k$ $(k\geqq 3)$ and $N\equiv 1 \pmod{p_i-1}$ for $i=1, 2, \cdots, k$.*

Therefore, Carmichael numbers are inherently composite numbers with three or more distinct prime factors. A family of Charmichael numbers with many prime factors are very interesting in connection with the following problem proposed by D. H. Lehmer [4], which remains still open.

Problem. *Does there exists a positive integer n such that the equation $n-1=k\phi(n)$ is satisfied for some integer $k>1$ where $\phi(n)$ is the Euler function?*

If there exists such a solution n, then n must be obviously a Carmichael number. D. H. Lehmer [4] showed that when $k=2$, this equation has no solution involving fewer than 7 distinct prime factors, and when $k=3$, a solution n is a product of more than 32 distinct prime factors. Later, many authors [3, 5, 6, 7, 10] discussed this problem especially about the lower bound of the number of distinct prime factors for any solution. At present time, it is known that such a solution n for $k=2$ (if exists) must have 13 or more distinct prime factors.

In the present note, we shall state several methods of constructing Carmichael numbers with many prime factors and we shall show some of the results so far obtained.

In our experiments, we have found about 300 Carmichael numbers with 13 or more prime factors, but we regret to say that we have not obtained any solution of Lehmer's problem.

All of computation of our experiments was done on a computer HITAC 20 in the Department of Mathematics, Okayama University.

1. Carmichael function. In his book [1], R. D. Carmichael has introduced a function $\lambda(n)$ which we call a Carmichael function. This function $\lambda(n)$ is defined in terms of Euler function $\phi(n)$ as follows:

$$\lambda(2^\alpha) = \phi(2^\alpha) \qquad \text{if } \alpha = 0,\ 1,\ 2,$$
$$\lambda(2^\alpha) = \frac{1}{2}\phi(2^\alpha) \qquad \text{if } \alpha > 2,$$
$$\lambda(p^\alpha) = \phi(p^\alpha) \qquad \text{if } p \text{ is an odd prime,}$$
$$\lambda(2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}) = M$$

where $M = \mathrm{LCM}\,(\lambda(2^\alpha),\ \lambda(p_1^{\alpha_1}),\ \lambda(p_2^{\alpha_2}), \cdots,\ \lambda(p_k^{\alpha_k}))$, $2,\ p_1,\ p_2, \cdots,\ p_k$ being different prime numbers.

By use of the function $\lambda(n)$, R. D. Carmichael proved the following

Theorem 2. (R. D. Carmichael [1]). *If a and n are relatively prime positive integers, then the congruence*

$$a^{\lambda(n)} \equiv 1 \pmod{n}$$

is satisfied.

This Carmichael's Theorem extends Fermat's general Theorem in the meaning of the fact that $\lambda(n)$ divides $\phi(n)$ properly except for the case where n is a power of a prime number. In particular, when an integer n is a square-free odd number, namely, $n = p_1 p_2 \cdots p_k$, then

$$\lambda(n) = \mathrm{LCM}\,(p_1 - 1,\ p_2 - 1, \cdots,\ p_k - 1).$$

Here, it is easily shown that if N is an integer such that $\lambda(N) \mid N-1$, then N must be a prime number or a square-free odd number consisting of three or more prime factors. Hence, we may restate Chernick's Theorem in terms of the Carmichael function as follows:

Theorem 3. *An integer N is a Carmichael number if and only if N is a composite number such that* $N \equiv 1 \pmod{\lambda(N)}$.

2. Product of two Carmichael numbers. Now, we consider certain conditions that the product of two relatively prime Carmichael numbers becomes again a Carmichael number.

Let N_1 and N_2 be distinct Carmichael numbers such that $(N_1, N_2) = 1$. Then, by Theorem 3,

$$\lambda(N_1) \mid N_1 - 1 \quad \text{and} \quad \lambda(N_2) \mid N_2 - 1$$

is satisfied.

On the other hand, from the expression

$$N_1N_2 - 1 = (N_1 - 1) + (N_2 - 1) + (N_1 - 1)(N_2 - 1),$$

if we suppose that $\lambda(N_1) \mid N_2 - 1$ and $\lambda(N_2) \mid N_1 - 1$ are true, then $N_1N_2 - 1$ is divisible by $\lambda(N_1)$ and $\lambda(N_2)$ respectively. From the fact that

$$\lambda(N_1N_2) = \text{LCM}\ (\lambda(N_1),\ \lambda(N_2)),$$

we have $\lambda(N_1N_2) \mid N_1N_2 - 1$. Whence, N_1N_2 is a Carmichael number. Thus, we have

Theorem 4. *Let N_1 and N_2 be a pair of Carmichael numbers satisfying the following conditions:*

(i) $(N_1, N_2) = 1$,
(ii) $\lambda(N_1) \mid N_2 - 1$,
(iii) $\lambda(N_2) \mid N_1 - 1$.

Then, the product N_1N_2 is a Carmichael number.

On applying the above Theorem 4 for seeking a new Carmichael number from two already known ones, this Theorem will not always be convenient, because it is seldom that one has such a pair of N_1 and N_2 satisfying the condition (i), (ii) and (iii). For the practical use, the following special cases of the above Theorem are more convenient.

Corollary 1. *Let N_1 and N_2 be Carmichael numbers. If the following conditions are satisfied*

(i) $(N_1, N_2) = 1$,
(ii) $\lambda(N_1) \mid \lambda(N_2)$ *and* $\lambda(N_2) \mid N_1 - 1$,

then the product N_1N_2 is a Carmichael number.

Corollary 2. *Let N_1 and N_2 be Carmichael numbers. If the following conditions are satisfied*

(i) $(N_1, N_2) = 1$,
(ii) $\lambda(N_1) = \lambda(N_2)$,

then, the product N_1N_2 is a Carmichael number.

Corollary 3 (J. Chenrnick [2]). *If N is a Carmichael number and p is an odd prime number satisfying the following conditions*

$$\text{(i)} \quad (N, p) = 1,$$
$$\text{(ii)} \quad \lambda(N) \mid p-1 \ \textit{and} \ p-1 \mid N-1,$$

then, the product Np is a Carmichael oumber.

Thus, fixing a certain Carmichael number N_1, we may seek such a Carmichael number N_2 from a list of Carmichael numbers.

In practice, if a table of Carmichael numbers classified by the values of Carmichael's λ-function is available, then we may seek more easily a pair of Carmichael numbers we desire. As an attempt, we have made such a tablet including about 2000 Carmichael numbers, but this extent is as yet insufficient.

3. Numerical examples.

1°. Take the following two Carmichael numbers.

$N_1 = 15841 = 7 \cdot 31 \cdot 73,$

and $N_2 = 340561 = 13 \cdot 17 \cdot 23 \cdot 67.$

Then, $\lambda(N_1) = 2^3 \cdot 3^2 \cdot 5 \mid N_2 - 1 = 2^4 \cdot 3^2 \cdot 5 \cdot 11 \cdot 43,$

$\lambda(N_2) = 2^4 \cdot 3 \cdot 11 \mid N_1 - 1 = 2^5 \cdot 3^2 \cdot 5 \cdot 11$

Whence, by Theorem 4,

$N_1 N_2 = 5394826801 = 7 \cdot 13 \cdot 17 \cdot 23 \cdot 31 \cdot 67 \cdot 73$

is a Carmichael number.

Other examples of the same nature are as follows:

$N_1 = 7 \cdot 31 \cdot 73 \qquad N_2 = 23 \cdot 199 \cdot 353,$
$N_1 = 7 \cdot 11 \cdot 13 \cdot 41 \qquad N_2 = 31 \cdot 61 \cdot 271,$
$N_1 = 13 \cdot 37 \cdot 241 \qquad N_2 = 31 \cdot 43 \cdot 61 \cdot 337,$
$N_1 = 13 \cdot 37 \cdot 241 \qquad N_2 = 61 \cdot 181 \cdot 5521,$
$N_1 = 19 \cdot 43 \cdot 409 \qquad N_2 = 109 \cdot 379 \cdot 919,$
$N_1 = 31 \cdot 61 \cdot 211 \qquad N_2 = 281 \cdot 421 \cdot 701.$

2°. Take the following two Carmichael numbers.

$N_1 = 2465 = 5 \cdot 17 \cdot 29,$

and $N_2 = 19384289 = 89 \cdot 353 \cdot 617$

Then, $\lambda(N_1) = 2^4 \cdot 7 \mid \lambda(N_2) = 2^5 \cdot 7 \cdot 11 \mid N_1 - 1 = 2^5 \cdot 7 \cdot 11.$

Whence, by Corollary 1,

$N_1 N_2 = 47782272385 = 5 \cdot 17 \cdot 29 \cdot 89 \cdot 353 \cdot 617$

is a Carmichael number.

Other examples of the same type are as follows:

$N_1 = 5 \cdot 29 \cdot 73 \qquad N_2 = 7 \cdot 19 \cdot 43 \cdot 113,$
$N_1 = 5 \cdot 29 \cdot 73 \qquad N_2 = 13 \cdot 17 \cdot 37 \cdot 113 \cdot 337,$

$N_1 = 7\cdot 11\cdot 13\cdot 41$ $\quad N_2 = 37\cdot 73\cdot 541$,
$N_1 = 13\cdot 37\cdot 241$ $\quad N_2 = 17\cdot 19\cdot 29\cdot 71\cdot 113$,
$N_1 = 13\cdot 37\cdot 241$ $\quad N_2 = 17\cdot 19\cdot 29\cdot 43\cdot 421$,
$N_1 = 13\cdot 17\cdot 41\cdot 61$ $\quad N_2 = 43\cdot 211\cdot 337$.

3°. Take the following two Carmichael numbers.

$N_1 = 512461 = 31\cdot 61\cdot 271$,

and $N_2 = 101957401 = 7\cdot 13\cdot 19\cdot 109\cdot 541$.

Then, $\lambda(N_1) = \lambda(N_2) = 540$.

Whence, by Corollary 2,

$$N_1 N_2 = 52249191673861 = 7\cdot 13\cdot 19\cdot 31\cdot 61\cdot 109\cdot 271\cdot 541$$

is a Carmichael number.

Other examples of the same type are as follows:

$N_1 = 11\cdot 37\cdot 113\cdot 631$ $\quad N_2 = 43\cdot 61\cdot 127\cdot 241$,
$N_1 = 13\cdot 37\cdot 241$ $\quad N_2 = 7\cdot 17\cdot 19\cdot 41\cdot 181$,
$N_1 = 31\cdot 37\cdot 43\cdot 181$ $\quad N_2 = 211\cdot 421\cdot 631$,
$N_1 = 11\cdot 31\cdot 61\cdot 521$ $\quad N_2 = 53\cdot 79\cdot 131\cdot 313$,
$N_1 = 43\cdot 211\cdot 337$ $\quad N_2 = 61\cdot 241\cdot 421$,
$N_1 = 13\cdot 17\cdot 661\cdot 881$ $\quad N_2 = 31\cdot 61\cdot 241\cdot 331$.

4. The two sieve methods. In the previous note [8], we have proposed a method which is fit to obtain Carmichael numbers with many prime factors. The principle of this method is as follows.

When $p_1 = 3$, $p_2 = 5, \cdots\cdots, p_t$ are consecutive odd prime numbers, we correspond for each square-free odd number $n = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ where $e_i = 0$ or 1 for $i = 1, 2, \cdots, t$, to a binary number $b = e_t e_{t-1} \cdots e_2 e_1$, and we sieve this binary number b by the principle due to the following

Theorem 5. *If N is a Carmichael number and p is one of prime factors of N, then, N does not contain any prime factor of the form $px + 1$.*

Thus. in principle, this binary sieve method*) is a method of removing such binary numbers containing simultaneously two binary digits corresponding to prime factors p and $px + 1$ respectively, from a sequence of consecutive binary numbers. In applying this method, we do not need necessarily to examine exhaustively consecutive binary numbers. If we always keep watch on the smallest bit number k such that $e_k = 1$, then we may examine only once the condition of the sieve about the kth bit e_k. When the present binary number $e_t e_{t-1} \cdots e_k 0 \cdots 0$ is to be sieved out, then

*) This nomenclature is given by Prof. S. Hitotsumatsu in Kyoto University.

we may examine the binary number $e_t e_{t-1} \cdots e_k 0 \cdots 0 + 2^{k-1}$ as a next one. On the other hand, when this binary number $e_t e_{t-1} \cdots e_k 0 \cdots 0$ passes through the sieve, after a series of processing, we must return to add 1 to the former binary number.

In the actual process, the above sieve can be performed skillfully by storing the table of the bit numbers to be sieved out for each prime number and the index of the above table in the memory.

In our experiments, we have adopted the following sieve method in addition. When $N = p_1 p_2 \cdots p_{m-1} p_m$ is a square-free odd number obtained by the above sieve, for each such N, we try to seek a prime number x for which Nx is a Carmichael number.

If p_{m-1} and p_m are two largest prime factors of N, then, by Chernick's Theorem, the congrunce $Nx \equiv 1 \pmod{\lambda(p_{m-1} p_m)}$ must hold. Since, by the construction of N, two integers N and $\lambda(p_{m-1} p_m)$ are relatively prime, this linear congruence has always a unique solution $x \equiv \bar{a} \pmod{\lambda(p_{m-1} p_m)}$. In the present case, this congruence is easily solvable on a computer as follows.

Firstly, we solve the congruence $Ny \equiv 1 \pmod{p_m - 1}$ which is simpler than the original one, by examining successively odd integers. When we obtain a solution $\bar{y}$ of this congruence, nextly, we try to seek a solution $\bar{a}$ of the form $\bar{a} = (p_m - 1) k + \bar{y}$ for $k = 0, 1, \cdots, \lambda(p_{m-1} p_m)/(p_m - 1)$ of the original congruence.

In the sequel, we may only seek prime numbers x such that $x \equiv \bar{a} \pmod{\lambda(p_{m-1} p_m)}$, from a list of prime numbers which are stored in the memory in advance. Then, the number Nx so obtained is a candidate of Carmichael numbers.

By use of these methods, we can save fairly many of multiplications and divisions from the computing processes in total.

5. Results. The results of our experiments are as in the following table. On the first several data, we appended the values of $N - 1$ and $2\phi(N)$ for each Carmichael number N. And others, we have listed only their prime factors. From the table, we have omitted the data which are listed up in the previous note [8].

1°. Carmichael numbers with 13 prime factors.

$N_1 = 5 \cdot 13 \cdot 17 \cdot 19 \cdot 29 \cdot 37 \cdot 43 \cdot 67 \cdot 89 \cdot 97 \cdot 113 \cdot 127 \cdot 317$

$N_1 - 1 = 254895\ 41690053\ 87850784$

$2\phi(N_1) = 291036\ 21553478\ 36141568$

$N_2 = 7 \cdot 11 \cdot 17 \cdot 19 \cdot 31 \cdot 37 \cdot 53 \cdot 61 \cdot 73 \cdot 79 \cdot 97 \cdot 131 \cdot 10369$

$N_2 - 1 = 7007969\ 86781114\ 74510080$
$2\phi(N_2) = 8462306\ 59984156\ 26240000$
$N_3 = 7 \cdot 13 \cdot 19 \cdot 23 \cdot 31 \cdot 37 \cdot 41 \cdot 61 \cdot 67 \cdot 89 \cdot 101 \cdot 103 \cdot 12241$
$N_3 - 1 = 8662440\ 10338475\ 74334800$
$2\phi(N_3) = 10717682\ 13226782\ 72000000$
$N_4 = 7 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 41 \cdot 47 \cdot 67 \cdot 73 \cdot 89 \cdot 97 \cdot 139 \cdot 1013$
$N_4 - 1 = 2272080\ 81796182\ 78887520$
$2\phi(N_4) = 2887796\ 00837763\ 66182400$
$N_5 = 7 \cdot 17 \cdot 19 \cdot 31 \cdot 37 \cdot 59 \cdot 67 \cdot 73 \cdot 83 \cdot 89 \cdot 97 \cdot 109 \cdot 577$
$N_5 - 1 = 3372529\ 82535163\ 18437120$
$2\phi(N_5) = 4433185\ 57382833\ 23555840$
$N_6 = 7 \cdot 17 \cdot 19 \cdot 37 \cdot 41 \cdot 47 \cdot 61 \cdot 67 \cdot 97 \cdot 101 \cdot 109 \cdot 139 \cdot 331$
$N_6 - 1 = 3237062\ 15801869\ 74136800$
$2\phi(N_6) = 4280330\ 05194313\ 72800000$

7·19·23·31·41·61·67·73·89·97·109·137·16831
11·13·17·19·29·37·41·61·71·73·113·127·20161
11·13·17·19·29·37·41·71·97·109·113·127·9721
11·13·17·19·31·37·41·43·71·97·109·127·3889
11·13·17·19·31·37·59·71·73·97·109·127·13921
11·13·17·29·31·37·43·61·71·73·97·109·8821
11·13·17·29·31·37·47·61·71·73·97·101·1201
11·13·17·31·37·41·43·61·71·73·109·113·281
11·13·17·31·41·43·61·71·73·101·113·127·2539
11·13·17·37·41·61·71·73·97·101·113·127·283
11·13·19·29·31·37·41·43·61·71·101·127·2801
11·13·19·29·31·37·41·43·61·73·113·127·379
11·13·19·29·31·37·43·61·71·73·97·127·18341
11·13·19·31·37·43·47·61·71·73·101·109·1657
11·13·19·31·37·43·61·73·97·103·113·127·8641
11·13·29·31·37·41·61·71·73·97·109·127·193
11·13·31·37·41·43·59·61·71·73·109·127·2437
11·13·31·37·47·61·71·73·97·103·113·127·6121
11·17·19·29·31·37·41·43·71·73·101·127·673
11·17·19·29·31·37·41·71·73·79·109·113·10531
11·17·19·29·37·41·43·61·71·73·97·113·2521
11·17·19·29·37·41·43·61·97·109·113·127·2141
11·17·19·29·41·43·61·71·73·101·109·113·701
11·17·19·37·41·53·61·71·73·79·97·113·421
11·17·31·37·41·43·61·71·73·101·113·127·211

11·17·31·37·41·53·71·73·79·97·109·113·2161
11·19·29·31·37·41·43·53·71·73·109·113·11467
11·19·29·31·37·41·43·61·73·79·103·113·7177
11·19·29·31·37·41·47·61·71·73·109·127·281
11·19·29·31·37·41·53·61·71·79·101·109·271
11·19·29·31·37·41·61·73·79·101·109·127·163
11·19·29·31·41·43·53·61·73·79·103·109·17341
11·19·31·37·41·43·53·61·71·73·79·113·1553
11·29·31·37·41·53·61·73·79·97·109·127·577
11·29·31·43·53·61·71·73·79·97·109·127·673
13·17·19·23·29·31·37·41·43·67·109·127·397
13·17·19·23·29·31·37·43·67·73·101·127·3301
13·17·19·23·29·31·41·43·89·109·113·127·379
13·17·19·23·29·31·41·61·67·73·89·113·10501
13·17·19·23·29·37·41·43·73·97·109·127·1009
13·17·19·23·31·37·41·43·61·73·109·113·433
13·17·19·23·31·37·41·67·71·73·97·127·1321
13·17·19·23·31·37·41·67·73·97·109·113·661
13·17·19·23·31·41·43·71·73·89·97·113·883
13·17·19·23·31·41·61·67·71·73·89·127·281
13·17·19·23·31·43·61·67·73·97·109·127·5281
13·17·19·23·31·61·67·73·89·97·113·127·631
13·17·19·23·41·43·71·73·89·97·107·127·1321
13·17·19·23·41·61·67·71·73·89·97·113·2377
13·17·19·29·31·37·41·61·73·89·109·113·617
13·17·19·29·31·37·41·67·73·109·113·127·3697
13·17·19·29·37·41·43·67·89·97·109·127·8317
13·17·19·29·37·41·43·71·73·97·113·127·701
13·17·19·29·37·41·61·73·97·109·113·127·3457
13·17·19·31·37·41·43·61·71·97·101·113·449
13·17·19·31·37·41·43·73·89·97·113·127·21001
13·17·23·29·31·37·41·43·73·89·113·127·1009
13·17·23·29·31·37·43·67·71·73·89·127·12101
13·17·23·29·31·37·61·67·71·73·97·127·331
13·17·23·29·31·41·43·61·67·89·101·113·2609
13·17·23·29·31·41·43·61·71·73·89·97·1009
13·17·23·29·31·41·43·73·89·109·113·127·2377
13·17·23·29·31·41·67·71·89·97·109·127·991
13·17·23·29·41·43·61·71·73·97·109·113·163
13·17·23·31·37·41·67·71·73·101·113·127·11551

13·17·23·31·37·43·61·67·71·73·113·127·397
13·17·29·31·37·43·67·71·73·97·109·113·1321
13·17·29·37·41·43·61·71·73·89·97·101·20161
13·17·29·41·43·61·67·71·89·109·113·127·12409
13·17·29·43·61·67·71·89·97·101·113·127·12553
13·17·31·37·41·43·61·71·73·89·97·127·7057
13·17·37·43·61·67·71·73·89·97·113·127·8317
13·19·23·29·31·37·41·43·61·73·89·97·2521
13·19·23·29·31·37·41·43·73·97·109·113·337
13·19·23·29·31·37·43·61·89·97·109·113·2269
13·19·23·29·31·41·43·61·67·109·113·127·181
13·19·23·31·37·41·43·61·67·73·109·127·8317
13·19·23·31·37·41·43·61·67·71·73·89·3169
13·19·23·31·37·41·67·71·73·103·107·113·9011
13·19·23·31·61·67·71·73·89·109·113·127·421
13·19·23·37·41·43·61·67·71·73·89·127·337
13·19·23·37·41·43·61·71·73·89·101·113·601
13·19·23·37·61·67·71·73·89·97·113·127·211
13·19·29·31·37·41·43·61·67·71·101·127·16633
13·19·29·31·37·43·61·67·71·97·103·109·19009
13·19·29·31·37·43·61·71·97·101·109·113·673
13·19·29·41·43·61·67·71·73·89·103·127·1321
13·19·29·41·43·61·67·73·89·103·109·113·661
13·19·31·37·41·61·67·73·89·97·101·109·12107
13·19·31·43·47·61·67·73·97·109·113·127·9721
13·19·37·41·43·61·71·73·89·97·101·127·21121
13·23·29·31·37·41·61·67·71·73·109·127·271
13·23·29·31·41·43·61·67·73·89·109·127·751
13·23·31·37·43·67·71·73·89·97·113·127·433
13·29·41·43·47·67·71·73·89·97·113·127·631
17·19·23·29·31·37·41·43·67·89·101·127·4201
17·19·23·29·31·43·53·61·71·79·89·109·241
17·19·23·29·31·61·67·71·73·79·97·113·8009
17·19·23·29·31·61·67·71·73·89·97·127·881
17·19·23·29·37·41·43·73·89·101·109·127·3361
17·19·23·29·37·41·61·67·71·79·89·127·21997
17·19·23·29·41·43·61·67·73·89·109·113·2689
17·19·23·31·37·41·43·61·71·73·109·127·421
17·19·23·31·37·41·61·67·71·79·101·109·2801
17·19·23·31·41·43·61·71·73·97·107·109·14741

17·19·23·31·41·43·71·73·89·109·113·127·541
17·19·23·37·41·67·71·73·89·97·109·113·12097
17·19·29·31·37·41·43·61·97·109·113·127·13441
17·19·29·31·37·41·53·61·79·97·109·127·8737
17·19·29·31·37·41·71·73·97·109·113·127·641
17·19·29·31·37·43·61·71·79·97·113·127·8461
17·19·29·31·37·43·61·73·79·89·97·113·21061
17·19·29·31·43·53·71·73·79·97·109·127·181
17·19·29·37·41·43·61·71·73·101·113·127·4201
17·19·31·37·41·43·61·71·97·109·113·127·379
17·19·31·37·43·67·71·73·89·101·109·127·12601
17·19·31·41·43·67·73·89·97·101·109·127·11251
17·19·37·41·43·53·67·73·79·89·109·127·1783
17·19·43·61·67·71·73·79·89·109·113·127·8581
17·23·29·31·37·41·61·67·73·89·113·127·16633
17·23·29·31·37·53·61·67·71·73·109·113·337
17·23·29·31·43·53·61·71·79·97·113·127·3697
17·23·29·37·41·43·53·67·73·79·101·127·19471
17·23·29·37·43·61·73·83·89·109·113·127·5281
17·23·29·41·43·61·67·71·89·97·101·113·421
17·23·31·41·53·71·73·79·89·97·101·113·1249
17·29·31·37·41·43·53·61·71·79·113·127·211
17·31·41·43·53·59·61·73·89·101·109·127·1873
19·23·29·31·37·43·67·73·89·103·113·127·1531
19·23·29·31·37·53·61·67·79·89·113·127·281
19·23·29·31·41·61·67·71·73·89·97·101·379
19·23·29·37·41·61·67·71·89·109·113·127·463
19·23·29·43·53·61·71·73·83·89·109·127·2341
19·23·29·43·61·67·71·73·89·97·109·127·241
19·23·31·37·41·61·67·73·89·101·103·127·3061
19·23·31·37·43·53·61·71·79·89·109·127·6007
19·23·31·41·43·53·61·71·89·97·101·127·157
19·29·31·37·43·53·67·71·79·109·113·127·1171
19·29·37·41·43·47·53·61·73·79·89·113·21529
23·29·31·37·41·61·67·73·79·89·97·127·4951
23·31·41·43·53·61·67·73·79·97·113·127·1321
31·37·41·61·67·73·89·97·101·103·113·127·337

2°. Carmichael numbers with 14 prime factors.

$N_1 = 5\cdot 13\cdot 17\cdot 19\cdot 37\cdot 67\cdot 73\cdot 89\cdot 97\cdot 109\cdot 113\cdot 127\cdot 139\cdot 2437$

$N_1 - 1 = 17\ 38024917\ 88293593\ 29894464$
$2\phi(N_1) = 20\ 47223472\ 69844179\ 23825664$
$N_2 = 5\cdot17\cdot19\cdot29\cdot37\cdot43\cdot53\cdot67\cdot79\cdot89\cdot97\cdot113\cdot127\cdot859$
$N_2 - 1 = 2\ 22461846\ 14735149\ 53150304$
$2\phi(N_2) = 2\ 67093881\ 80363730\ 13921792$
$N_3 = 7\cdot13\cdot19\cdot23\cdot31\cdot41\cdot61\cdot73\cdot89\cdot97\cdot103\cdot107\cdot137\cdot4241$
$N_3 - 1 = 12\ 44207050\ 15914998\ 28496800$
$2\phi(N_3) = 15\ 56995574\ 14996436\ 58240000$
$N_4 = 7\cdot13\cdot19\cdot31\cdot37\cdot41\cdot61\cdot73\cdot89\cdot97\cdot109\cdot137\cdot139\cdot4049$
$N_4 - 1 = 26\ 27037946\ 60544116\ 52521920$
$2\phi(N_4) = 33\ 53039349\ 49017601\ 96608000$
$N_5 = 7\cdot19\cdot23\cdot37\cdot41\cdot53\cdot61\cdot73\cdot79\cdot89\cdot101\cdot103\cdot131\cdot2161$
$N_5 - 1 = 22\ 67751074\ 88494393\ 11342800$
$2\phi(N_5) = 30\ 22042478\ 45036359\ 68000000$
$N_6 = 11\cdot13\cdot17\cdot19\cdot29\cdot31\cdot37\cdot41\cdot43\cdot61\cdot71\cdot113\cdot127\cdot8821$
$N_6 - 1 = 1\ 48504788\ 68674230\ 84533040$
$2\phi(N_6) = 1\ 83569802\ 78645227\ 52000000$

11·13·17·19·29·31·41·43·61·71·73·97·127·641
11·13·17·19·29·31·41·43·61·73·97·101·127·251
11·13·17·19·29·31·41·47·71·101·109·113·139·421
11·13·17·19·31·43·59·61·71·73·97·127·139·3709
11·13·17·19·37·43·47·61·71·73·97·109·139·2017
11·13·17·29·31·37·43·47·61·71·97·113·127·2521
11·13·19·29·31·37·43·61·71·73·109·113·127·3001
11·13·19·31·37·41·43·61·71·73·103·109·113·3541
11·13·19·31·43·71·73·103·109·113·127·137·139·337
11·13·29·37·41·43·61·71·97·101·109·113·127·2161
11·17·19·29·31·37·41·43·61·73·113·127·131·181
11·17·19·29·31·37·41·43·71·73·97·109·127·281
11·17·19·29·31·37·41·61·71·73·97·101·109·6301
11·17·19·29·31·37·47·53·61·71·73·79·113·449
11·17·19·29·37·41·43·47·53·61·73·127·139·937
11·17·19·31·37·41·53·61·73·79·113·127·131·3121
11·17·19·31·37·43·61·79·97·109·113·127·131·8971
11·17·19·31·41·43·47·53·61·71·113·127·131·7489
11·17·29·31·37·43·53·61·71·73·79·127·131·4051
11·17·29·37·43·61·71·79·83·101·113·127·131·15991
11·19·29·31·37·43·61·71·73·107·113·127·137·18523
11·19·29·31·37·53·61·71·73·79·109·131·137·2161

11·19·29·31·41·43·53·61·71·73·79·103·127·12377
11·19·29·31·41·43·53·79·97·109·113·127·137·15121
11·19·29·31·43·53·61·73·97·109·113·127·131·641
11·19·29·37·41·71·73·97·103·109·127·131·137·3361
11·19·37·41·61·71·79·97·101·103·113·127·137·1549
11·29·31·37·41·43·53·61·71·79·97·113·127·2521
11·29·37·61·71·73·79·97·101·103·113·127·131·1801
13·17·19·23·29·31·37·41·43·61·89·97·113·673
13·17·19·23·29·31·37·41·43·67·71·97·113·20161
13·17·19·23·29·31·37·41·43·71·97·109·127·9857
13·17·19·23·29·31·37·41·43·73·89·113·127·2971
13·17·19·23·29·31·37·43·61·67·89·107·109·2333
13·17·19·23·29·31·41·43·67·71·73·89·109·7129
13·17·19·23·31·37·67·71·89·97·101·113·127·3709
13·17·19·23·37·41·43·61·71·73·97·109·127·4231
13·17·19·23·37·41·61·71·73·89·97·113·127·3361
13·17·19·29·31·37·41·43·61·73·97·113·127·2521
13·17·19·29·31·37·41·61·67·71·73·97·113·1913
13·17·19·29·31·37·43·61·71·73·109·113·127·163
13·17·19·29·31·37·47·61·67·71·73·97·127·3361
13·17·19·29·31·43·61·67·71·73·89·109·127·4861
13·17·19·37·41·61·67·71·89·97·101·109·113·1051
13·17·23·29·31·37·41·43·67·71·97·113·127·397
13·17·23·31·37·41·43·61·67·71·101·113·127·1801
13·17·29·31·37·41·43·61·67·71·89·97·113·2113
13·17·31·37·41·43·47·67·71·73·89·97·127·16193
13·19·23·29·31·37·41·43·71·73·89·107·127·463
13·19·23·29·31·37·41·61·67·71·101·109·127·617
13·19·23·29·31·41·43·61·67·89·103·109·127·10099
13·19·23·29·31·43·61·71·73·89·101·113·127·7129
13·19·23·29·31·43·67·71·73·89·97·103·127·661
13·19·29·37·41·43·61·67·71·73·97·109·113·16633
13·23·29·31·41·43·61·71·73·89·97·101·113·701
13·23·29·37·41·43·61·67·71·73·89·103·113·631
17·19·23·29·31·41·43·61·67·79·89·113·127·331
17·19·23·29·31·43·53·61·73·79·97·113·127·11621
17·19·23·31·41·43·61·67·73·97·109·113·127·661
17·19·23·31·43·61·67·73·79·97·101·109·113·3433
17·19·23·37·41·43·53·61·71·73·79·89·127·1093
17·19·29·31·37·41·53·61·71·97·109·113·127·1801

17·19·29·31·37·43·47·53·67·79·89·109·113·3037
17·19·31·37·41·43·53·59·67·71·79·89·127·13729
17·19·31·37·41·43·61·71·73·79·109·113·127·17389
17·23·29·37·43·53·61·67·73·89·97·113·127·2281
17·23·31·37·41·43·53·67·71·73·89·109·127·181
17·23·31·41·43·53·61·67·71·79·89·97·101·2861
17·23·31·37·41·53·59·61·67·73·79·89·109·991
17·31·37·43·53·61·67·71·73·79·97·113·127·157
19·29·31·41·43·61·71·73·97·101·103·113·127·10949
19·29·37·41·43·53·61·67·73·97·103·109·113·15121
19·37·41·43·61·67·71·79·89·97·101·113·127·281
23·29·31·37·43·61·67·71·89·97·107·113·127·6679
23·29·37·41·43·53·61·67·71·79·89·97·127·8737

3°. Carmichael numbers with 15 prime factors.

$N_1 = 11·13·17·19·29·31·41·43·61·71·73·109·113·127·181$
$N_1 - 1 = 6\ 55313092\ 67520060\ 31481760$
$2\phi(N_1) = 8\ 09205661\ 26272839\ 68000000$
$N_2 = 11·13·17·29·37·43·61·71·73·97·101·109·113·127·3361$
$N_2 - 1 = 1826\ 56218859\ 96899086\ 02922400$
$2\phi(N_2) = 2416\ 82757497\ 13488117\ 76000000$
$N_3 = 11·13·19·29·31·37·41·47·61·71·73·97·113·139·829$
$N_3 - 1 = 69\ 54458522\ 68659408\ 93948960$
$2\phi(N_3) = 89\ 30254222\ 75157950\ 46400000$
$N_4 = 11·13·19·29·31·37·41·61·71·73·103·109·127·139·967$
$N_4 - 1 = 224\ 51981346\ 43216749\ 88335360$
$2\phi(N_4) = 292\ 38695806\ 08100368\ 38400000$
$N_5 = 11·17·19·29·31·37·41·61·73·97·101·109·113·127·421$
$N_5 - 1 = 139\ 21211814\ 25387264\ 34990400$
$2\phi(N_5) = 184\ 96129400\ 29093478\ 40000000$
$N_6 = 11·17·19·29·31·37·43·53·61·71·73·79·113·131·911$
$N_6 - 1 = 90\ 72086866\ 72895691\ 33969120$
$2\phi(N_6) = 118\ 88746343\ 27406673\ 92000000$

11·17·19·29·31·37·43·53·61·71·79·101·113·131·8443
11·17·19·29·37·41·43·47·71·73·101·109·113·139·241
11·17·19·37·41·47·73·79·97·107·113·127·131·139·2969
11·17·29·37·41·47·53·71·73·79·101·109·127·139·17389
11·17·31·41·43·47·53·61·71·79·101·113·127·139·241
11·19·29·31·37·41·47·61·71·79·97·127·131·139·8971

11·19·29·31·37·47·53·61·71·73·79·113·131·139·5297
11·19·31·37·41·43·47·73·79·97·103·127·131·139·3457
11·19·31·37·41·43·61·71·73·79·101·109·113·127·443
11·29·31·37·41·43·61·71·73·79·97·113·127·131·151
11·29·37·43·47·53·71·73·97·101·113·127·131·139·17551
13·17·19·23·29·31·37·41·43·61·89·101·109·127·991
13·17·19·23·29·31·37·43·61·71·73·89·113·127·1321
13·17·19·23·31·37·41·43·67·71·73·89·101·127·4951
13·17·19·29·31·37·41·43·61·89·101·109·113·127·7561
13·17·19·29·31·41·43·47·61·71·97·109·113·127·337
13·19·23·29·31·43·61·67·71·73·89·103·127·137·1361
13·19·23·29·37·41·61·67·71·89·97·103·127·137·5281
13·19·37·41·43·61·67·71·73·97·101·109·113·127·271
17·19·23·29·31·37·41·43·53·61·67·71·73·127·331
17·19·23·29·31·37·41·43·71·73·89·97·101·113·8443
17·19·29·31·37·41·43·61·71·79·97·101·113·127·8641
17·19·29·31·41·53·61·67·71·73·89·97·109·113·2801
17·19·31·37·41·43·53·67·71·73·89·109·113·127·1009
17·23·31·37·41·43·61·67·71·73·89·109·113·127·15661
17·29·31·37·41·43·53·61·67·73·89·97·101·127·6469
19·23·29·31·37·43·53·67·71·79·89·97·113·127·1801
19·23·31·37·43·53·61·71·73·79·89·97·103·127·661
23·29·31·37·41·53·61·71·73·79·89·97·101·113·661

4°. Carmichael numbers with 16 prime factors.

$N_1 = 11 \cdot 17 \cdot 19 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 53 \cdot 61 \cdot 71 \cdot 73 \cdot 79 \cdot 109 \cdot 127 \cdot 3511$
$N_1 - 1 =$ 13405 52999385 33956373 34499040
$2\phi(N_1) =$ 17143 31093587 27806320 64000000
$N_2 = 11 \cdot 17 \cdot 19 \cdot 29 \cdot 41 \cdot 43 \cdot 53 \cdot 61 \cdot 71 \cdot 73 \cdot 79 \cdot 97 \cdot 101 \cdot 113 \cdot 131 \cdot 13001$
$N_2 - 1 =$ 453397 50707544 50828588 20476000
$2\phi(N_2) =$ 603872 83013455 57708800 00000000
$N_3 = 11 \cdot 19 \cdot 29 \cdot 31 \cdot 37 \cdot 43 \cdot 53 \cdot 61 \cdot 71 \cdot 73 \cdot 97 \cdot 103 \cdot 113 \cdot 127 \cdot 131 \cdot 409$
$N_3 - 1 =$ 38481 21695695 92503606 14291200
$2\phi(N_3) =$ 52696 55160231 05568374 78400000

11·19·29·37·41·47·53·61·71·79·97·101·103·131·137·911
13·19·29·31·37·41·43·61·71·73·103·109·113·137·139·5153
17·19·23·29·31·37·41·43·61·67·71·73·79·97·113·199
17·19·23·29·31·37·41·43·61·67·71·73·89·97·127·991
17·19·23·29·37·41·43·53·61·71·79·97·109·113·131·337

17·19·29·31·37·41·43·47·53·61·73·97·109·127·131·601
17·19·23·31·37·41·43·67·71·79·89·97·113·127·131·20411
17·19·23·31·37·43·53·61·67·71·73·79·109·113·131·18481
17·19·31·37·41·43·53·67·73·79·89·97·113·127·131·6007
17·19·37·41·43·53·61·67·71·73·79·89·101·109·131·20021
17·19·37·43·53·61·67·73·79·89·101·109·113·127·131·8581
17·23·31·37·43·53·61·67·71·73·79·89·97·113·127·11971

5°. Carmichael numbers with 17 prime factors.

N_1 = 13·17·19·23·29·31·37·41·43·61·67·71·73·97·113·127·211
$N_1 - 1$ = 35237 86921171 88895473 10642240
$2\phi(N_1)$ = 43865 42048572 99809337 34400000
N_2 = 17·19·23·29·37·41·53·61·67·71·73·79·89·97·113·127·1153
$N_2 - 1$ = 4140720 88758206 70901469 40775680
$2\phi(N_2)$ = 5680470 49546689 54683776 69632000
N_3 = 17·19·23·31·37·43·61·71·73·79·89·97·109·113·127·131·2521
$N_3 - 1$ = 40814540 51159215 43476961 10175720
$2\phi(N_3)$ = 57187975 33611025 80001800 19200000

17·19·23·31·37·43·61·71·73·79·89·97·109·113·127·131·2521
17·19·29·37·41·53·61·67·71·73·79·97·101·113·127·131·18481
19·29·31·37·43·53·61·73·89·97·101·103·113·127·131·137·353
19·31·37·41·43·53·67·71·73·79·89·97·109·113·127·131·8581

6°. Carmichael numbers with 18 prime factors.

N_1=17·19·23·29·31·37·41·43·61·73·79·89·101·109·113·127·131·1783
$N_1 - 1$ = 5 03345922 12408193 81421431 01799600
$2\phi(N_1)$ = 6 74001137 88987089 78592645 12000000
N_2=19·23·29·37·41·61·67·71·73·79·89·101·103·113·127·131·137·4421
$N_2 - 1$ = 339 17286568 02790349 93171667 12003600
$2\phi(N_2)$ = 492 08709048 79762731 52416153 60000000

REFERENCES

[1] R. D. CARMICHAEL: The theory of numbers and Diophantine analysis. Dover Publ. Inc. (1915).

[2] J. CHERNICK: On Fermat's simple theorem. Bul. Amer. Math. Soc. **45** (1938), 268—274.

[3] M. KISHORE: On the equation $k\phi(M) = M-1$. Not. Amer. Math. Soc. **22** (1975), A—501—A—502.

[4] D. H. LEHMER: On Euler's totient function. Bul. Amer. Math. Soc. **38** (1932), 745—751.

[5] E. LIEUWENS: Do there exist composite numbers M for which $k\phi(M) = M-1$ holds? Nieuw Arch. Wisk. (3), **18** (1970), 165—169.

[6] C. POMERANCE: On composite n for which $\phi(n) \mid n-1$, II. Paciffic J. Math. **69** (1977), 177—186.

[7] F. SHUH: Can $n-1$ be divisible by $\phi(n)$ when n is composite? Math. Zutphen. B. **12** (1944), 102—107.

[8] M. YORINAGA: Numerical computation of Carmichael numbers. Math. J. Okayama Univ. Vol. **20** (1978), 151—163.

[9] M. YORINAGA: Numerical computation of Carmichael numbers. II. Math. J. Okayama Univ. Vol. **21** (1979), 183—205.

[10] S.S. WAGSTAFF, JR.: Large Carmichael numbers. Math. J. Okayama Univ. Vol. **22** (1980), 33—41.

DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY

(*Received March 27, 1980*)