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ON VON NEUMANN REGULAR RINGS. V

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Introduction. For several years, von Neumann regular rings and related rings are extensively studied (cf. for example, the bibliographies of [3] and [5], while for rings without identity, consult [6], [15], [16]). This paper is motivated by a question of Goodearl [5, Problem 10] concerning unit-regular rings and the question raised in [21]. Semi-simple Artinian rings and strongly regular rings are well-known examples of unit-regular rings, but arbitrary von Neumann regular rings need not be unit-regular. The class of unit-regular rings is closed under homomorphic images, direct products and direct limits. Such rings have many interesting properties (cf. [5]). Certain regular rings, V-rings and associated rings are here proved to be unit-regular. For example, left and right V-rings whose essential left ideals are ideals are unit-regular. (This is related to [5, Problem 10] and extends [5, Corollary 4. 2].) Such rings, if indecomposable, are simple Artinian. But we first continue the study of ALD rings (redefined below) introduced in [21]. Further properties of ALD rings are developed, and certain results in [21] will be improved. A positive answer is given to the question raised in [21, Remark (p. 340)] which is related to [3, Query (a)].

Throughout, A represents an associative ring with identity and A-modules are unitary. We follow the notations and definitions in [6], [12] to [21]. Since for a given proper essential left ideal E of A, any maximal left subideal is either essential in or a direct summand of $_AE$ (in the latter case, it is a complement left subideal), the definition of ALD rings in [21] may be reformulated as follows: A is called an *ALD* (*almost left duo*) *ring* if, for any proper essential left ideal E of A, every complement left subideal is an ideal of E and E is an ideal of A.

In [21, Proposition 2. 1], it is proved that a simple left module over an ALD ring is injective iff it is p-injective. Consequently, ALD regular rings are left V-rings. It is then natural to ask whether ALD left V-rings are von Neumann regular [21, Remark]. This is answered in the first section. Results in [5], [10] and [21] are generalized. Next, the decompositions of certain p-injective and p-V-rings will yield unit regular rings and new characteristic properties of semi-simple Artinian rings.

1. Von Neumann regular rings. We first prove an important lemma for ALD rings.

Lemma 1.1. *If A is a semi-prime ALD ring, then A is either semi-simple Artinian or reduced.*

Proof. Assume that A is not semi-simple Artinian. Then there exists a maximal left ideal M of A which is essential in ${}_AA$. Suppose there exists non-zero $a \in A$ such that $a^2 = 0$. If $l(a) \subseteq M$, then $a \in M$. If $l(a) \not\subseteq M$, then $M + l(a) = A$ and $1 = b + c$, where $b \in M$, $c \in l(a)$, which implies $a = ba \in M$ (an ideal of A). Thus $a \in M$ in any case. By Zorn's lemma, there exists a complement left subideal K of M such that $Aa \oplus K$ is essential in ${}_AM$. Then $KM \subseteq K$ implies $Ka \subseteq K \cap Aa = 0$, and hence $Aa \oplus K \subseteq l(a)$. Since $l(a)$ is then an ideal, we have $(Aa)^2 = A(aA)a \subseteq Al(a)a = 0$, contradicting the semi-primeness of A. This proves the lemma.

With suitable modifications, the proof of [14, Lemma 3], [4, Theorem 2. 38] and Lemma 1. 1 yield

Proposition 1. 2. *Let A be a semi-prime ALD ring.*
(1) *The maximal left quotient ring of A coincides with the right one.* (This partly extends Utumi's result (cf. [5, Theorem 3. 8]).)
(2) *If A is left or right continuous, then A is either semi-simple Artinian or a left and right continuous strongly regular ring.*

Remark 1. Since flat modules play an important role in ring theory, the following may be noted: For any p-injective left ideal I of A, A/I is a flat left A-module.

Following [12], a left A-module M is called *semi-simple if* $J(M)$, the Jacobson radical of M, is zero. The next result, which is related to [3, Query (a)], imporoves [21, Proposition 2. 1 (3), (4), Corollary 2. 2, Theorems 2. 4 and 2. 5], while answering at the same time the question raised in [21, Remark].

Theorem 1. 3. *The following conditions are equivalent for an ALD ring A:*

1) *A is von Neumann regular.*
2) *A is a left V-ring.*
3) *A is a right V-ring.*
4) *A is fully idempotent.*
5) *$E = E^*$ for every essential left subideal E of any proper principal left ideal of A.*

6) *Every cyclic semi-simple left A-module is flat.*

7) *Every maximal essential right ideal of A is p-injective.*

8) *Every simple right A-module is flat.*

9) *A is a semi-prime ring such that A/P is regular for every completely prime ideal P of A.*

10) *A is a semi-prime left or right p-injective ring.*

Proof. 1) implies 2) and 3) by Lemma 1. 1, while 2) implies 5) by [12, Theorem 2. 1].

3) implies 1) by [20, Propsition 9].

1) implies 4) and 6) through 10) evidently.

4) $\Longrightarrow$ 1): Note that a reduced ring is fully left idempotent iff fully right idempotent. Then [20, Proposition 9] and [21, Proposition 2. 1 (4)] prove the implication.

5) $\Longrightarrow$ 2): Since the Jacobson radical of A is zero, every minimal left ideal of A is injective [21, Lemma 1. 1]. Suppose there exists a simple left A-module V which is not injective. Then for any proper principal left ideal P of A and any non-zero left A-homomorphism $f: P \to V$, $K = \mathrm{Ker} f$ is a maximal left subideal of P which is necessarily essential in ${}_AP$, whence $K = K^*$. Then there exists a maximal left ideal M of A such that $K \subseteq M$ but $P \not\subseteq M$. Since $M \cap P = K$ and $M + P = A$, then $A/K = P/K \oplus M/K$, which shows that f may be extended to $g: {}_AA \to {}_AV$. Therefore ${}_AV$ is p-injective, which implies that ${}_AV$ is injective by [21, Proposition 2. 1 (1)]. This contradiction proves that 5) implies 2).

6) $\Longrightarrow$ 1): Since $J(A/J(A)) = 0$, $A/J(A)$ is a flat left A-module. For any $b \in J(A)$, $b = bc$ for some $c \in J(A)$ and there exists $d \in A$ such that $(1-c)d = 1$. Then $b = b(1-c)d = 0$. Thus $J(A) = 0$. Since every simple left A-module is flat, A is regular by [19, Theorem 1. 4] and Lemma 1. 1.

7) implies 8) by Remark 1.

8) implies 1) by [19, Theorem 1. 4 and Lemma 2. 1] and Lemma 1. 1.

9) implies 1) by [5, Theorem 1. 21] and Lemma 1. 1.

10) implies 1) by [7, Theorem 1], [16, Theorem 5] and Lemma 1. 1.

Remark 2. (1) [5, Corollary 1. 18] and Theorem 1. 3 4) imply that if every prime factor ring of A is ALD, then A is regular iff A is fully idempotent.

(2) An ALD left V-ring is either semi-simple Artinian or strongly regular and therefore unit-regular (cf. [19, Question]).

Theorem 1. 4. *The following are equivalent:*

1) *A is a prime ALD ring.*
2) *A is either simple Artinian or a left duo, left Ore domain.*

Proof. 1) $\Longrightarrow$ 2): Suppose A is not simple Artinian. By Lemma 1.1, A is reduced and hence an integral domain. If C is a non-zero complement left ideal of A then there exists a left ideal D such that $C \oplus D$ is essential in ${}_AA$. Since A is an integral domain, it is easy to see that $D=0$. This proves $C=A$. Noting that any left ideal of A is essential in a complement left ideal, we readily see that A is a left duo, left Ore domain.

2) $\Longrightarrow$ 1): Obvious.

Call A an *ELT ring* if every essential left ideal is an ideal of A. It is known that prime ELT left self-injective rings are simple Artinian [8]. We may add the following to [21, Theorem 2.8].

Theorem 1.5. *The following conditions are equivalent:*
1) *A is simple Artinian.*
2) *A is a prime ALD left V-ring.*
3) *A is a prime ALD ring with a divisible simple left module.*
4) *A is a prime ELT left and right V-ring.*

Proof. Obviously, 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3) and 1) $\Longrightarrow$ 4).

3) $\Longrightarrow$ 1): If A is not simple Artinian, then A is a left duo, left Ore domain by Theorem 1.4. Let U be a simple left A-module which is divisible. Then U is p-injective, since A is an integral domain. If $U \simeq A/M$, then M is a maximal essential left ideal of A. For any non-zero $b \in M$, we consider the left A-homomorphism $f: Ab \to A/M$ defined by $f(ab) = a + M$ for all $a \in A$. Then there exists $c \in A$ such that $1 + M = f(b) = bc + M$. Since $bc \in M$, we obtain $1 \in M$, which contradicts $M \neq A$.

4) $\Longrightarrow$ 1): By a remark in [5, Problem 52 (p. 350)], [20, Proposition 9] and Proposition 2.3 below.

Corollary 1.6. *Let A be a fully idempotent ring whose prime factor rings are ALD. Then A is a unit-regular left and right V-ring. In that case, A is left self-injective iff A is right self-injective.*

Proof. Apply [5, Theorem 6.10 and Proposition 6.18] to Remark 2 (1) and Theorem 1.5. The last part follows from [5, Corollary 6.22].

We now consider strongly regular rings. The next result contains improvements of some of the equivalent conditions in [10, Theorem].

Theorem 1.7. *The following conditions are equivalent:*

1) *A is strongly regular.*

2) *A is a left and right duo ring such that $L \cap R = LR$ for every essential left ideal L and every essential right ideal R of A.*

3) *A is either a left or right duo ring such that $L \cap R = RL$ for every essential left ideal L and every essential right ideal R of A.*

4) *A is a left duo ring such that $L_1 \cap L_2 = L_1L_2$ for all essential left ideals L_1, L_2 of A.*

5) *$P \cap L = PL$ for every principal left ideal P and every essential left ideal L of A.*

6) *A is an ELT fully left idempotent ring such that every proper prime ideal is completely prime.*

7) *Every maximal left ideal of A is an ideal and every simple right A-module is flat.*

8) *Every maximal left ideal I of A is an ideal and A/I_A is flat.*

Proof. By [1, Remark (1) (p. 248)], A is fully idempotent iff $I^2 = I$ for any essential ideal I of A. It therefore follows that if A is a left duo ring such that $I^2 = I$ for every essential ideal I of A, then A is strongly regular. It is then easy to see that 1), 2) and 4) are equivalent.

Obviously, 1) implies 3), 5) through 8).

3) $\Longrightarrow$ 1): For any $b \in A$, there exists a left ideal K and a right ideal R such that $L = Ab \oplus K$ is essential in ${}_AA$ and $E = bA \oplus R$ is essential in A_A. Then $b \in EL$ yields $b \in bAb$.

5) $\Longrightarrow$ 1): Obviously, A is a left duo ring. For any $b \in A$, there exists a left ideal K such that $L = Ab \oplus K$ is essential in ${}_AA$. Then $b \in AbL$ yields $b \in Ab^2$.

6) $\Longrightarrow$ 1): If P is a proper prime ideal of A, then A/P is an ELT, fully left idempotent integral domain, and therefore a division ring. Hence A is strongly regular by [5, Corollary 1.18 and Theorem 3.2].

7) $\Longrightarrow$ 8): If I is a maximal left ideal, then A/I is a division ring, and therefore A/I_A is flat by 7).

8) $\Longrightarrow$ 1): It suffices to show that $Ab + l(b) = A$ for every $b \in A$. Suppose $Ac + l(c) \neq A$ for some $c \in A$. If L is a maximal left ideal containing $Ac + l(c)$, then A/L_A is flat and hence $Ay \cap L = Ly$ for any $y \in A$, in particular $c = dc$ with some $d \in L$. Then, $1 = (1-d) + d \in l(c) + L = L$, which is a contradiction.

2. ELT rings. A well-known theorem of Jain-Mohamed-Singh [9, Theorem 2. 3] states that A is an ELT left self-injective ring iff every left ideal of A is quasi-injective. We begin this section with a lemma which contains direct consequences of definitions.

Lemma 2. 1. *Let $A=B\oplus C$, where B, C are ideals of A.*

(1) *If A is an ELT ring then both B and C are ELT rings.*

(2) *Every left C-module is a left A-module, and a left C-module which is a p-injective left A-module is a p-injective left C-module.*

(3) *If A is a left p-injective ring then both B and C are left p-injective rings.*

The next decomposition theorem is motivated by [20, Question (p. 128)]. Following [15], we say that A is a *left p-V-ring* if every simple left A-module is p-injective. Throughout, S denotes the left socle of A.

Theorem 2. 2. *The following conditions are equivalent for a ring A:*

1) *A is a direct sum of a semi-simple Artinian ring and a strongly regular ring with zero socle.*

2) *A is an ELT left p-V-ring with finitely generated left socle.*

3) *A is a semi-prime ELT ring whose simple right modules are flat and such that ${}_AS$ is finitely generated.*

4) *A is a semi-prime ELT left p-injective ring such that ${}_AS$ is finitely generated.*

Proof. It is easy to see that 1) implies 2) through 4).

2)$\Longrightarrow$1): Since a left p-V-ring is fully left idempotent, A is a semi-prime ring such that ${}_AS$ is finitely generated. Then it is known that $S=Ae$ for some central idempotent e. Let $T=A(1-e)$. Then $A=S\oplus T$, S is semi-simple Artinian, and T is an ELT left p-V-ring with zero socle (Lemma 2. 1 (1), (2)). Since T/I_T is flat for any ideal I of T, the condition 8) of Theorem 1. 7 is satisfied. Hence, T is strongly regular.

3)$\Longrightarrow$1): Again $A=S\oplus T$, where S is a semi-simple Artinian ring and T is an ELT ring with zero socle such that every simple right T-module is flat. Then T is strongly regular by Theorem 1. 7 7).

4)$\Longrightarrow$1): We obtain $A=S\oplus T$, where S is a semi-simple Artinian ring and T is a semi-prime ELT left p-injective ring with zero socle (Lemma 2. 1 (1), (2), (3)). Since T is semi-prime ELT, by applying [19, Lemma 2. 1], we see that T is left non-singular, and $J(T)=0$ [9, p. 213]. Let M be an arbitrary maximal left ideal of T, and let $t\in T$ such that $t^2=0$. If $l_T(t)\not\subseteq M$ then $M+l_T(t)=T$ implies $t\in M$ (an ideal of T).

Thus $t \in M$ in any case, and $t \in J(T) = 0$, which proves that T is reduced. Now, T is strongly regular by [7, Theorem 1] and [16, Theorem 5].

In view of Theorem 2. 2, we raise the following

Question 1. Suppose that A is a semi-prime ELT ring satisfying any one of the following conditions: 1) A is left p-injective; 2) every simple right A-module is flat. Then, is A von Neumann regular?

It is now known that a prime regular ring need not be primitive [2]. However, the proof of Theorem 2. 2 and [19, Theorem 1. 4] imply the following:

Proposition 2. 3. *A prime ELT ring is primitive with non-zero socle if A satisfies any one of the following conditions*: 1) *A is left p-injective*; 2) *A is fully left or right idempotent*; 3) *every cyclic semi-simple left A-module is flat* (cf. [3, Problem 3]).

Remark 3. (1) If A is a prime ring with non-zero left singular ideal, then every non-zero left ideal of A contains a non-zero nilpotent element.

(2) A is a primitive left self-injective regular ring with non-zero socle iff A is a prime left self-injective ring with a maximal right annihilator.

In [16, Theorem 6], semi-simple Artinian rings are characterized in terms of ELT left Goldie rings. We here give a few characteristic properties in terms of ELT rings with maximum condition on annihilators.

Theorem 2. 4. *The following conditions are equivalent*:

1) *A is semi-simple Artinian.*

2) *A is an ELT left V-ring with maximum conditionon left annihilators.*

3) *A is an ELT left V-ring with maximum condition on right annihilators.*

4) *A is an ELT left p-V-ring without infinite sets of orthogonal idempotents.*

5) *A is an ELT left and right V-ring whose proper factor rings satisfy the maximum condition on left annihilators.*

Proof. Apply [17, Proposition 3], [19, Lemma 1. 2], Theorem 2. 2

and [13, Corollary 1].

We now return to unit-regular rings and consider the following question of Goodearl [5, Problem 10 (p. 344)]: Are regular left and right V-rings unit-regular? A particular answer is contained in the next theorem.

Theorem 2.5. *An ELT left and right V-ring is unit-regular.*

Proof. Since every factor ring of an ELT ring is ELT, it is sufficient to apply [5, Theorem 6.10], [20, Proposition 9] and Theorem 1.5 4).

The next is a combination of [8, Theorem 2.3], [20, Proposition 9], Theorems 2.2 and 2.5.

Corollary 2.6. *A is unit-regular in each of the following cases:*

1) *A is an ELT continuous right V-ring.*
2) *A is a right V-ring whose essential left ideals are quasi-injective.*
3) *A is an ELT right V-ring whose minimal left ideals are injective.*
4) *A is an ELT left p-V-ring such that ${}_AS$ is finitely generated.*
5) *A is a semi-prime ELT left p-injective ring such that ${}_AS$ is finitely generated.*

As usual, $M_n(A)$ denotes the ring of all $n \times n$ matrices over A. For direct finiteness, consult [5, Chapter 5]. Applying [5, Proposition 6.11, Corollaries 4.7, 6.4, 6.12, 6.16, and Theorems 4.14, 6.6, 6.21] to Theorems 1.5 4) and 2.5, we get

Proposition 2.7. *Let A be an ELT left and right V-ring.*

(1) *The maximal left quotient ring of A coincides with the right one.*

(2) *Every non-zero ideal of A contains a non-zero central idempotent.*

(3) *If F is a finitely generated left A-module, then every injecive or surjective endomorphism of F is an automorphism, and F is directly finite.*

(4) *Let P be a finitely generated projective left A-module. Then* (a) $\mathrm{End}_A(P)$ *is a unit-regular left and right V-ring;* (b) *if M, N are left A-modules such that $P \oplus M \simeq P \oplus N$, then $M \simeq N$.*

(5) *If M, N are finitely generated projective left A-modules and n is a positive integer such that the direct sum of n copies of M is isomorphic to the direct sum of n copies of N, then $M \simeq N$.*

(6) *If B is an ELT left and right V-ring and n is a positive integer such that $M_n(A) \simeq M_n(B)$, then $A \simeq B$.*

Proposition 2.7 (2) implies the following

Corollary 2.8. *An indecomposable ELT left and right V-ring is simple Artinian.*

Finally, [5, Corollary 6.3 and Theorem 6.10], Theorem 1.5 4), Theorem 2.5 and Proposition 2.7 (1) yield the following answer to [3, Query (c)].

Corollary 2.9. *Let A be a regular ring whose maximal left quotient ring Q is an ELT, right V-ring. Then Q is right self-injective and A is a unit-regular left and right V-ring.*

In [11], it is shown that certain results on injective and p-injective modules have analogues in the theory of semi-groups. We conclude with the following:

Question 2. Are there semi-group analogues of Theorems 1.3 and 2.4?

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