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COSEPARABLE COALGEBRAS AND COEXTENSIONS OF CODERIVATIONS

Dedicated to Professor Yoshikazu Nakai on his sixtieth birthday

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Throughout the present paper, k will be a fixed field. All vector spaces are k-vector spaces and linear maps are k-linear. Unadorned $\otimes$ means $\otimes_k$. As for notations and terminologies used here, we follow [4] and [5].

Let A and C be coalgebras, and $\phi: A \longrightarrow C$ a coalgebra map. If M is an A-comodule, then M is a C-comodule via ϕ. Let M be an A-A-bicomodule. A C-comodule map $f: M \longrightarrow A$ is called a *C-coderivation* if

$$\Delta_A f = (1 \square f)\rho^- + (f \square 1)\rho^+ : M \longrightarrow A \square_C A,$$

where ρ^- (resp. ρ^+) is the left (resp. right) A-comodule structure map of M. A C-coderivation f is an *inner C-coderivation* if there exists a C-comodule map $\gamma: M \longrightarrow C$ such that

$$f = (1 \square \varepsilon_C \gamma)\rho^- - (\varepsilon_C \gamma \square 1)\rho^+ .$$

This is a generalization of the notion of a k-coderivation in the sense of Doi [2]. A C-coderivation $\tau: M \longrightarrow A$ is called a *coextension* of a k-coderivation $\delta: M \longrightarrow C$ if $\phi\tau = \delta$.

In what follows, we assume always that C is a cocommutative coalgebra, A is a C-coalgebra (i. e., a coalgebra over C via ϕ ([4, p. 127])), and that M is an A-A-bicomodule.

One of the purposes of this paper is to extend Doi's theorem on coseparable coalgebras [2, Th. 3] to coseparable C-coalgebras. We prove also the following: If A is a C-injective coalgebra and if $H^2(N, A) = 0$ for all A-A-bicomodules N (in the sense of Jonah [3, §4]), then for any k-coderivation $\delta: M \longrightarrow C$, there exists a C-coderivation $\tau: M \longrightarrow A$ which is a coextension of δ. Note that if A is a coseparable coalgebra, then $H^2(N, A) = 0$.

1. Coseparable coalgebras. In this section, we extend Doi's theorem on coseparable coalgebras [2, Th. 3] to our coseparable C-coalgebras. First, we consider the following exact sequence of A-A-bicomodules

$$(1.1) \qquad 0 \longrightarrow A \xrightarrow{\Delta_A} A \square_C A \xrightarrow{\omega} (A \square_C A)/\Delta_A(A) \longrightarrow 0$$

where ω is the canonical linear map and the A-A-bicomodule structure of $(A\square_C A)/\Delta_A(A)$ is defined naturally. We set $L=(A\square_C A)/\Delta_A(A)$ and $a\circ b=\omega(a\square b)$.

Lemma 1.1 (cf. [2, § 3]). *A linear map* $\lambda: L\longrightarrow A$ *defined by*

$$\lambda(a\circ b)=a\varepsilon_C\phi(b)-\varepsilon_C\phi(a)b$$

is a C-coderivation.

Proof. By the definition of C-coalgebras and the A-A-bicomodule structure of L, we obtain $\Delta_A\lambda=(1\square\lambda)\rho_L^-+(\lambda\square 1)\rho_L^+$, where ρ_L^- (resp. ρ_L^+) is the right (resp. left) A-comodule structure map of L. It remains to show that λ is a left C-comodule map. We have

$$(\phi\otimes 1)\Delta_A\lambda(a\circ b)=\textstyle\sum_{(a)}\phi(a_{(1)})\otimes a_{(2)}\varepsilon_C\phi(b)-\sum_{(b)}\varepsilon_C\phi(a)\phi(b_{(1)})\otimes b_{(2)}$$

and

$$\begin{aligned}(1\otimes\lambda)(\phi\otimes 1)\rho_L^-(a\circ b)&=(1\otimes\lambda)(\textstyle\sum_{(a)}\phi(a_{(1)})\otimes a_{(2)}\circ b)\\&=\textstyle\sum_{(a)}\phi(a_{(1)})\otimes a_{(2)}\varepsilon_C\phi(b)-\phi(a)\otimes b.\end{aligned}$$

Since $a\circ b$ is in L, we have

$$\textstyle\sum_{(a)}a_{(1)}\otimes\phi(a_{(2)})\otimes b=\sum_{(b)}a\otimes\phi(b_{(1)})\otimes b_{(2)},$$

and therefore

$$\textstyle\sum_{(b)}\varepsilon_C\phi(a)\phi(b_{(1)})\otimes b_{(2)}=\sum_{(a)}\varepsilon_C\phi(a_{(1)})\otimes\phi(a_{(2)})\otimes b=\phi(a)\otimes b.$$

Hence $(\phi\otimes 1)\Delta_A\lambda=(1\otimes\lambda)(\phi\otimes 1)\rho_L^-$, which shows that λ is a left C-comodule map. By the cocommutativity of C, λ is a C-comodule map.

Theorem 1.2. *Let A be a C-coalgebra. Then the following conditions are equivalent:*

(a) *A is a coseparable C-coalgebra.*

(b) *For any A-A-bicomodule M, every C-coderivation from M to A is an inner C-coderivation.*

Proof. (a)$\Longrightarrow$(b). Since A is a coseparable C-coalgebra, there exists a linear map $\tau: A\square_C A\longrightarrow A$ such that $\tau\Delta_A=1$ and $\Delta_A\tau=(1\otimes\tau)(\Delta_A\square 1)=(\tau\otimes 1)(1\square\Delta_A)$. Hence we have

$$\tau=(1\otimes\varepsilon_A\tau)(\Delta_A\tau\square 1)=(\varepsilon_A\otimes 1)(1\square\Delta_A).$$

Let $f: M \longrightarrow A$ be an arbitrary C-coderivation. Setting $h = \phi\tau(1\square f)\rho^-$: $M \longrightarrow C$, we can easily see that h is a C-comodule map. By the property of τ, we have

$$(1\square \varepsilon_C h)\rho^- = (1\square \varepsilon_C\phi)\Delta_A\tau(1\square f)\rho^- = \tau(1\square f)\rho^-.$$

Since $f = \tau\Delta_A f = \tau(1\square f)\rho^- + \tau(f\square 1)\rho^+$ and $\varepsilon_C\phi f = \varepsilon_A f = 0$, we obtain that

$$(\varepsilon_C h\square 1)\rho^+ = (\varepsilon_C\phi\square 1)((f-\tau(f\square 1)\rho^+)\square 1)\rho^+ = -(\varepsilon_C\phi\square 1)\Delta_A\tau(f\square 1)\rho^+$$
$$= -\tau(f\square 1)\rho^+.$$

Therefore $f = (1\square\varepsilon_C h)\rho^- - (\varepsilon_C h\square 1)\rho^+$, which shows that f is an inner C-coderivation.

(b) $\Longrightarrow$ (a). This can be proved by the same way as in the proof (iv) $\Longrightarrow$ (i) of [2, Th. 3]. For the sake of comleteness, we give the proof. By assumption, there exists a C-comodule map $\gamma: L \longrightarrow C$ such that $\lambda = (1\square\varepsilon_C\gamma)\rho_L^- - (\varepsilon_C\gamma\square 1)\rho_L^+$. Define $\xi: L \longrightarrow A\square_C A$ by $\xi = (1\square\varepsilon_C\square 1)(1\square\gamma\square 1)(\rho_L^-\square 1)\rho_L^+$. Then it is easy to see that ξ is an A-A-bicomodule map and

$$\xi = (1\square\varepsilon_C\square 1)((\lambda\square 1 + (\gamma\square 1)\rho_L^+\square 1)\rho_L^+$$
$$= (1\square\varepsilon_C\square 1)((\lambda\square 1)\rho_L^+ + (1\square\Delta_A)(\gamma\square 1)\rho_L^+) = (\lambda\square 1)\rho_L^+ + \Delta_A(\gamma\square 1)\rho_L^+.$$

By $\omega\Delta_A = 0$, we have $\omega\xi = \omega(\lambda\square 1)\rho_L^+$. Finally, we shall show that $\omega(\lambda\square 1)\rho_L^+ = 1$. If $a\circ b$ is in L, then

$$\omega(\lambda\square 1)\rho_L^+(a\circ b) = \omega((\textstyle\sum_{(b)} a\varepsilon_C\phi(b_{(1)}) - \varepsilon_C\phi(a)b_{(1)})\otimes b_{(2)})$$
$$= a\circ b - \omega(\textstyle\sum_{(b)}\varepsilon_C\phi(a)b_{(1)}\otimes b_{(2)}) = a\circ b.$$

Thus the sequence (1. 1) is split as A-A-bicomodule.

2. Coextensions of coderivations. Let B be the direct sum of A and M as a vector space.

In [3], Jonah shows the following: Let $\Delta_B: B \longrightarrow B\otimes B$ and ε_B: $B \longrightarrow k$ be linear maps defined respectively by

$$\Delta_B = \begin{pmatrix} \Delta_A & 0 \\ 0 & \rho^- \\ 0 & \rho^+ \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \varepsilon_B = \begin{pmatrix} \varepsilon_A \\ 0 \end{pmatrix}.$$

Then $(B, \Delta_B, \varepsilon_B)$ is a coalgebra.

Now, let $\delta: M \longrightarrow C$ be a k-coderivation, and let $\rho_B: B \longrightarrow C\otimes B$

be a linear map defined by

$$\rho_B = \begin{pmatrix} (\phi\otimes 1)\Delta_A & (\delta\otimes 1)\rho^+ \\ 0 & (\phi\otimes 1)\rho^- \end{pmatrix}.$$

We shall show that (B, ρ_B) is a left C-comodule. Since δ is a k-coderivation and M is a C-comodule, we have $\Delta_C\delta = (1\otimes\delta)(\phi\otimes 1)\rho^- + (\delta\otimes 1)(1\otimes\phi)\rho^+$, and so

(2. 1) $(\Delta_C\delta\otimes 1)\rho^+ = (\phi\otimes\delta\otimes 1)(1\otimes\rho^-)\rho^- + (\delta\otimes\phi\otimes 1)(1\otimes\Delta_A)\rho^+$.

Moreover it is easy to see that

$$(\Delta_C\otimes 1)\rho_B = \begin{pmatrix} (\Delta_C\phi\otimes 1)\Delta_A & (\Delta_C\otimes 1)(\delta\otimes 1)\rho^+ \\ 0 & (\Delta_C\otimes 1)(\phi\otimes 1)\rho^- \end{pmatrix}$$

and

$$(1\otimes\rho_B)\rho_B = \begin{pmatrix} (\phi\otimes\phi\otimes 1)(1\otimes\Delta_A)\Delta_A & (\delta\otimes\phi\otimes 1)(1\otimes\Delta_A)\rho^+ + (\phi\otimes\delta\otimes 1)(1\otimes\rho^+)\rho^- \\ 0 & (\phi\otimes\phi\otimes 1)(1\otimes\rho^-)\rho^- \end{pmatrix}$$

Then by (2. 1), we have $(\Delta_B\otimes 1)\rho_C = (1\otimes\rho_B)\rho_B$ and

$$(\varepsilon_C\otimes 1)\rho_B = \begin{pmatrix} (\varepsilon_C\otimes 1)(\phi\otimes 1)\Delta_A & (\varepsilon_C\otimes 1)(\delta\otimes 1)\rho^+ \\ 0 & (\varepsilon_C\otimes 1)(\phi\otimes 1)\rho^- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus B is a left C-comodule.

Finally, by making use of the structure of $A\oplus M$ mentioned above, we shall prove the following

Theorem 2.1. *Let A be a C-coalgebra, and let $\delta: M\longrightarrow C$ be a k-coderivation. If A is an injective C-comodule and if $H^2(N, A) = 0$ for all A-A-bicomodules N, then there exists a C-coderivation $\tilde{\delta}: M\longrightarrow A$ which is a coextension of δ.*

Proof. As is claimed above, $B = A\oplus M$ is a coalgebra and a C-comodule. Since the canonical projection $B\longrightarrow A$ is a coalgebra map, B is a C-coalgebra. Consider the exact sequence of C-comodules

(2. 2) $$0\longrightarrow A\xrightarrow{i} B\xrightarrow{p} M\longrightarrow 0$$

where i is the canonical injection and p is the canonical projection. Then (2. 2) is a singular coalgebra extension ([3, §4]). By the C-injectivity of A, (2. 2) is split as C-comodule. Hence by [3, Th. 4. 10], there exists a C-coalgebra map $\tilde{\delta}: B\longrightarrow A$ such that $\tilde{\delta}i = 1$. Identifying m in M with $\begin{pmatrix} 0 \\ m \end{pmatrix}$ in B, $\Delta_A\tilde{\delta} = (\tilde{\delta}\otimes\tilde{\delta})\Delta_B$ implies

$$\Delta_A\tilde{\delta}(m) = (1 \otimes \tilde{\delta})\rho^-(m) + (\tilde{\delta} \otimes 1)\rho^+(m). \tag{2. 3}$$

Thus $\tilde{\delta}$ is a C-coderivation. Moreover, since $\tilde{\delta}$ is a C-comodule map we have

$$(\tilde{\phi} \otimes 1)\Delta_A\tilde{\delta} = (1 \otimes \tilde{\delta})\begin{pmatrix} (\phi \otimes 1)\Delta_A & (\delta \otimes 1)\rho^+ \\ 0 & (\phi \otimes 1)\rho^- \end{pmatrix}$$

and so by (2. 3), we obtain

$$((\phi \otimes \tilde{\delta})\rho^+ + (\phi\tilde{\delta} \otimes 1)\rho^-)(m) = ((\delta \otimes 1)\rho^+ + (\phi \otimes \tilde{\delta})\rho^-)(m).$$

Therefore $(\phi\tilde{\delta} \otimes 1)\rho^+ = (\delta \otimes 1)\rho^+$ on M. This shows that $\phi\tilde{\delta} = \delta$.

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