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ON MODIFIED CHAIN CONDITIONS

To Professor Yoshikazu Nakai on his sixtieth birthday

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Throughout the present paper, A will represent a ring without (possibly with) identity, N the prime radical of A, and M a left A-module. Given a left ideal I of A and an A-submodule M' of M, for each positive integer i we set $I^{-i}M' = \{u \in M \mid I^i u \subseteq M'\}$. Following F. S. Cater [1], we say that M is *almost Artinian* (resp. *almost Noetherian*) if for each infinite descending (resp. ascending) chain $M_1 \supseteq M_2 \supseteq \cdots$ (resp. $M_1 \subseteq M_2 \subseteq \cdots$) of A-submodules of M there exist positive integers m, q such that $A^q M_m \subseteq M_i$ (resp. $M_i \subseteq A^{-q}M_m$) for all i, or equivalently there exists a positive integer p such that $A^p M_p \subseteq M_i$ (resp. $M_i \subseteq A^{-p}M_p$) for all i. Every left A-module which is Artinian (resp. Noetherian) in the usual sense is clearly almost Artinian (resp. almost Noetherian). If ${}_AA$ is almost Artinian (resp. almost Noetherian), we say that A is an *almost left Artinian* (resp. *almost left Noetherian*) ring.

If M is a trivial left A-module, i. e. $AM = 0$, then clearly M is both almost Artinian and almost Noetherian. Further, every nilpotent ring is both almost left Artinian and almost left Noetherian. It is easy to construct a nilpotent ring which is neither left Artinian nor left Noetherian, e. g. $\begin{pmatrix} 0 & 0 \\ \boldsymbol{Q} & 0 \end{pmatrix}$ is such a ring. On the other hand, $\begin{pmatrix} \boldsymbol{Q} & 0 \\ \boldsymbol{Q} & 0 \end{pmatrix}$ is a non-nilpotent ring which is almost left Artinian but not left Artinian, and $\begin{pmatrix} \boldsymbol{Z} & 0 \\ \boldsymbol{Q} & 0 \end{pmatrix}$ is a non-nilpotent ring which is almost left Noetherian but neither left Noetherian nor almost left Artinian.

In §1, several preliminary results in [1] will be reproved with notable briefness. In §2, we shall improve Theorems A, B of [1] (Theorems 1 and 2). The principal theorem of §3 states that if A is almost left Noetherian then A satisfies the ascending chain condition for semiprime ideals, every nil subring of A is nilpotent and the nilpotency indices of nil subrings are bounded (Theorem 3). In §4, we shall give some new conditions for a ring to be almost left Artinian (Theorem 4).

1. We begin with improving Propositions 4 and 9 of [1] all together.

Proposition 1. (1) *The following are equivalent*:

1) ${}_AM$ *is almost Artinian.*

2) *For each infinite descending chain* $M_1 \supseteq M_2 \supseteq \cdots$ *of A-submodules of M there exists a positive integer* q *such that* $A^q M_q = A^q M_i$ *for all* $i > q$.

3) *In each non-empty family* $\mathscr{M}$ *of A-submodules of M such that* $M' \in \mathscr{M}$ *implies* $AM' \in \mathscr{M}$, *there exists a minimal member.*

4) *For each non-empty family* $\mathscr{M}$ *of A-submodules of M, there exists a positive integer q and a member* M' *of* $\mathscr{M}$ *such that* $A^q M' \subseteq M''$ *for every* $M'' \in \mathscr{M}$ *with* $M'' \subseteq M'$.

(2) *The following are equivalent:*

1) ${}_AM$ *is almost Noetherian.*

2) *For each infinite ascending chain* $M_1 \subseteq M_2 \subseteq \cdots$ *of A-submodules of M there exists a positive integer* q *such that* $A^{-q} M_q = A^{-q} M_i$ *for all* $i > q$.

3) *In each non-empty family* $\mathscr{M}$ *of A-submodules of M such that* $M' \in \mathscr{M}$ *implies* $A^{-1}M' \in \mathscr{M}$, *there exists a maximal member.*

4) *For each non-empty family* $\mathscr{M}$ *of A-submodules of M, there exists a positive integer q and a member* M' *of* $\mathscr{M}$ *such that* $M'' \subseteq A^{-q} M'$ *for every* $M'' \in \mathscr{M}$ *with* $M' \subseteq M''$.

Proof. (1) As is easily seen, 4) $\Longrightarrow$ 3) $\Longrightarrow$ 2) $\Longrightarrow$ 1). Now, suppose 4) does not hold for some $\mathscr{M}$. Then we can find successively $M_i \in \mathscr{M}$ $(i = 1, 2, \cdots)$ such that $M_{i+1} \subset M_i$ but $A^i M_i \not\subseteq M_{i+1}$. We have thus seen that 1) implies 4).

(2) Obviously, 4) $\Longrightarrow$ 3) $\Longrightarrow$ 2) $\Longrightarrow$ 1). Suppose now that 4) does not hold for some $\mathscr{M}$. Then we can find successively $M_i \in \mathscr{M}$ $(i = 1, 2, \cdots)$ such that $M_i \subset M_{i+1}$ but $M_{i+1} \not\subseteq A^{-i} M_i$. Thus we have seen that 1) implies 4).

Now, Proposition 1 makes short the proof of [1, Proposition 7].

Proposition 2 ([1, Proposition 7]). (1) *Let* M' *be an A-submodule of M. Then* ${}_AM$ *is almost Artinian if and only if both* ${}_AM'$ *and* ${}_AM/M'$ *are almost Artinian.*

(2) *Let* M' *be an A-submodule of M. Then* ${}_AM$ *is almost Noetherian if and only if both* ${}_AM'$ *and* ${}_AM/M'$ *are almost Noetherian.*

Proof. (1) It suffices to prove the if part. Let $M_1 \supseteq M_2 \supseteq \cdots$ be an arbitrary descending chain of A-submodules of M. By Proposition 1 (1), there exists a positive integer p such that $A^p M_p + M' = A^p M_i + M'$ and $A^p(M_p \cap M') = A^p(M_i \cap M')$ for all $i > p$. Since $A^p M_p \subseteq A^p M_i + (A^p M_p \cap M') \subseteq A^p M^i + (M_p \cap M')$, it follows that $A^{2p} M_p \subseteq A^{2p} M_i +$

$A^p(M_p \cap M') = A^{2p}M_i + A^p(M_i \cap M') \subseteq M_i$.

(2) It is enough to prove the if part. Let $M_1 \subseteq M_2 \subseteq \cdots$ be an arbitrary ascending chain of A-submodules of M. There exists a positive integer p snch that $A^pM_i + M' \subseteq M_p + M'$ and $A^p(M_i \cap M') \subseteq M_p \cap M'$ for all i. Since $A^pM_i \subseteq M_p + (M_i \cap M')$, it follows $A^{2p}M_i \subseteq A^pM_p + A^p(M_i \cap M') \subseteq M_p$.

A left A-module M is said to be *s-unital* if $u \in Au$ for each $u \in M$, or equivalently if $M' = AM'$ for each A-submodule M' of M. If ${}_AA$ is s-unital, we term A a left s-unital ring. Any ring A with a left identity is a left s-unital ring. Obviously, for s-unital left A-modules, the concept of "almost Artinian" (resp. "almost Noetherian") coincides with that of "Artinian" (resp. "Noetherian"). Now, suppose that $A/\mathrm{Ann}(M)$ is left s-unital. Then by [6, Theorem 1], ${}_AAM$ is seen to be s-unital, and therefore by Proposition 2 (1) (resp. (2)), ${}_AM$ is almost Artinian (resp. almost Noetherian) when and only when ${}_AAM$ is Artinian (resp. Noetherian). In particular, if $A/l(A)$ is left s-unital, then A is almost left Artinian (resp. almost left Noetherian) when and only when A^2 is a left Artinian (resp. Noetherian) ring.

Lemma 1. (1) *If a unital left A-module M is almost Artinian, then the socle of ${}_AM$ is essential in ${}_AM$.*

(2) *If a left A-module M is the sum of s-unital A-submodules M_λ ($\lambda \in \Lambda$), then M is s-unital. In particular, every completely reducible left A-module is s-unital.*

Proof. (1) Immediate from the condition 3) of Proposition 1 (1).

(2) Let u be an arbitrary element of M. Then $u = u_1 + \cdots + u_k$ with some $u_i \in M_{\lambda_i}$. If $k = 1$ then $au = u$ with some $a \in A$, by hypothesis. Now, assume $k > 1$, and choose $b \in A$ such that $bu_k = u_k$. Then $u - bu = (u_1 - bu_1) + \cdots + (u_{k-1} - bu_{k-1})$. By induction method, there exists $c \in A$ such that $c(u - bu) = u - bu$. We conclude then $u = (b + c - cb)u$.

The next is [1, Lemma 2]. However, for the sake of convenience, we shall give a somewhat economical proof.

Lemma 2. *Let A be an almost left Artinian ring.*

(1) *Every non-nilpotent left ideal contains a minimal non-nilpotent left ideal.*

(2) *Every nil left ideal of A is nilpotent.*

Proof. (1) is obvious by the condition 3) of Proposition 1 (1). In order to prove (2), suppose contrarily that there exists a nil left ideal I which is not nilpotent. By (1), we may assume that I is a minimal non-nilpotent left ideal. Consider the family of all left subideals I' of I with $II' \neq 0$. Then, again by the condition 3) of Proposition 1 (1), the family contains a minimal member I^*. Since $II^* = I^*$, there exists $a^* \in I^*$ such that $Ia^* = I^*$. Hence, $aa^* = a^*$ ($\neq 0$) with some $a \in I$. Obviously, a is not nilpotent. But this contradicts the hypothesis that I is nil.

Now, by making use of Lemmas 1 and 2, we reprove [1, Theorem 1].

Proposition 3. *If A is almost left Artinian, then A is semiprimary, namely N is nilpotent and A/N is Artinian (semisimple).*

Proof. Since N is nilpotent by Lemma 2 (2), it suffices to prove that if A is semiprime and almost left Artinian then A is Artinian semisimple. By Lemma 1 (1), the left socle S of A is essential in ${}_AA$. Since ${}_AS$ is completely reducible and Artinian (Lemma 1 (2)) and every minimal left ideal of A is generated by an idempotent, it is known that S itself is generated by an idempotent. Hence, S coincides with A, whence we can conclude the assertion.

2. First, we state the following that includes Theorems A and B of [1].

Theorem 1. *Let $I, I_1, \cdots, I_k$ be left ideals of A.*

(1) *If ${}_AA/I$ is completely reducible and $IM = 0$, then the following are equivalent:*

1) *${}_AM$ is almost Artinian.*
2) *${}_AAM$ is Artinian.*
3) *${}_AAM$ is finitely generated.*
4) *${}_AAM$ is Noetherian.*
5) *${}_AM$ is almost Noetherian.*

(2) *If ${}_AA/I_j$ $(j = 1, \cdots, k)$ are completely reducible and $I_1 \cdots I_k M = 0$, then the following are equivalent:*

1) *${}_AM$ is almost Artinian.*
2) *${}_A(AM/I_kM)$, ${}_A(AI_kM/I_{k-1}I_kM)$, $\cdots$, ${}_A(AI_2 \cdots I_kM/I_1I_2 \cdots I_kM)$ are finitely generated.*
3) *${}_AM$ is almost Noetherian.*

In particular, if A is semiprimary then a left A-module is almost Artinian *if and only if it is almost Noetherian.*

Proof. (1) It is easy to see that ${}_A AM$ is completely reducible. Hence, the equivalence of 2), 3) and 4) is obvious. Since ${}_A(A/\mathrm{Ann}(M))$ is s-unital by Lemma 1 (2), the equivalences of 1) and 2) and of 4) and 5) are evident by the remark mentioned just before Lemma 1.

(2) Observe the descending chain

$$M \supseteq I_k M \supseteq I_{k-1} I_k M \supseteq \cdots \supseteq I_2 \cdots I_k M \supseteq I_1 \cdots I_k M = 0.$$

Then the assertion can be proved by (1) and Proposition 2 (1).

Now, let Λ_M be the set of almost Artinian A-submodules of M, and Γ_M the set of A-submodules U of M such that ${}_A M/U$ is almost Noetherian. Obviously, Λ_M and Γ_M contain 0 and M, respectively. Moreover, by Proposition 2 (1) (resp. (2)), if M' and M'' are in Λ_M (resp. Γ_M) then $M' + M''$ and $A^{-1}M'$ (resp. $M' \cap M''$ and AM') are in Λ_M (resp. Γ_M). We set $\Lambda(M) = \sum_{U \in \Lambda_M} U$ and $\Gamma(M) = \bigcap_{U \in \Gamma_M} U$. Needless to say, if ${}_A M$ is almost Artinian (resp. almost Noetherian) then $\Lambda(M) = M$ (resp. $\Gamma(M) = 0$), but not conversely. If ${}_A M$ is almost Noetherian, then by Proposition 1 (2) we see that $\Lambda(M)$ is the greatest member of Λ_M and is characterized as the least one among the A-submodules U of M with $\Lambda(M/U) = 0$; in particular ${}_A M$ is almost Artinian if and only if $\Lambda(M) = M$. On the other hand, if ${}_A M$ is almost Artinian, then by Proposition 1 (1) we see that $\Gamma(M)$ is the least member of Γ_M and is characterized as the greatest one among the A-submodules U of M with $\Gamma(U) = U$; in particular ${}_A M$ is almost Noetherian if and only if $\Gamma(M) = 0$.

In the proof of the following partial extension of Theorem 1 (1), we shall use freely the facts mentioned above.

Theorem 2. *Let I be a left ideal of A such that ${}_A A/I$ is completely reducible.*

(1) *If ${}_A M$ is almost Noetherian and $I^{-1}M' \neq M'$ for every proper A-submodule M' of M, then ${}_A M$ is almost Artinian.*

(2) *If ${}_A M$ is almost Artinian and $IM' \neq M'$ for every non-zero A-submodule M' of M, then ${}_A M$ is almost Noetherian.*

Proof. (1) Suppose $\Lambda(M) \neq M$, and choose an A-submodule $M'' \supset \Lambda(M)$ such that $IM'' \subseteq \Lambda(M)$. Since $\Lambda(M/\Lambda(M)) = 0$, we see that $A(M''/\Lambda(M)) \neq 0$. Then, ${}_A A(M''/\Lambda(M))$ is completely reducible and Noetherian (Lemma 1), and therefore Artinian. This is a contradiction. Thus ${}_A M$ is almost Artinian.

(2) Obviously, $A\Gamma(M)=\Gamma(M)$. Now, suppose $\Gamma(M)\neq 0$. Then ${}_A(\Gamma(M)/I\Gamma(M))$ is completely reducible and Artinian (Lemma 1), and therefore Noetherian. This contradiction means that ${}_AM$ is almost Noetherian.

3. In this section, we shall prove the following:

Theorem 3. *Let A be an almost left Noetherian ring.*

(1) *A satisfies the ascending chain condition for semiprime ideals.*

(2) *Every nil subring of A is nilpotent and the nilpotency indices of nil subrings are bounded.*

In preparation for the proof, we establish the next lemma.

Lemma 3. *Let A be an almost left Noetherian ring. If $r(A)=0$ (in particular, if A is semiprime), then A is a left Goldie ring.*

Proof. Let $L_1\subseteq L_2\subseteq\cdots$ be an infinite ascending chain of left annihilators, where $L_i=l(S_i)$. Then there exists a positive integer p such that $A^pL_i\subseteq L_p$ for all i. Since $A^pL_iS_p=0$, it follows $L_iS_p=0$, namely $L_i\subseteq L_p$. Next, assume that A contains an infinite direct sum of non-zero left ideals $I_1\oplus I_2\oplus\cdots$. There exists a positive integer q such that

$$A^q(I_1\oplus\cdots\oplus I_i)\subseteq I_1\oplus\cdots\oplus I_q \text{ for all } i.$$

Then $A^qI_i=0$, and therefore $I_i=0$ for all $i>q$. But, this is a contradiction.

Proof of Theorem 3. (1) The proof is straightforward.

(2) There exists a positive integer q such that $A^pr(A^i)\subseteq r(A^q)$ for all i. Since $A^{2q}r(A^i)\subseteq A^qr(A^q)=0$, there holds $r(A^i)\subseteq r(A^{2q})$. This means that the right annihilator of $A/r(A^{2q})$ is zero. Hence, $A/r(A^{2q})$ is a left Goldie ring by Lemma 3. According to [2, Corollary 1. 7], there exists a positive integer n such that $K^n\subseteq r(A^{2q})$ for all nil subrings K of A. It is immediate that $K^{2q+n}=0$.

Combining Theorem 3 (2) with Proposition 3 and the latter part of Theorem 1 (2), we readily obtain

Corollary 1. *If A is almost left Artinian, then every nil subring of A is nilpotent and the nilpotency indices of nil subrings are bounded.*

4. In advance of stating the main theorem of this section, we shall prove the following

Lemma 4. (1) *If A is almost left Artinian, then A is a π-regular ring of bounded index.*

(2) *Let A be an almost left Noetherian, π-regular ring. If A/N is left s-unital, then A/N is Artinian.*

Proof. (1) By Proposition 3 and [5, Lemma 2].

(2) By Lemma 3, A/N is a left Goldie ring. Then, as was claimed in the proof of [6, Theorem 3], A/N contains the identity. Moreover, it is easy to see that every regular element of A/N is a unit. Hence, A/N coincides with its left quotient ring that is Artinian semisimple.

A left ideal I of A is said to be *almost maximal* if A/I is a sum of minimal A-submodules. If a prime ideal P is an almost maximal left ideal, then ${}_A A/P$ is completely reducible. (In [1], a prime ideal is called an almost prime ideal.)

We are now ready to complete the proof of our main theorem, which includes Theorems 5, 6 and 11 of [1].

Theorem 4. *The following are equivalent:*

1) *A is almost left Artinian.*

2) *N is nilpotent and ${}_A(AN^{i-1}/N^i)$ is Artinian for all $i>0$.*

3) *A is almost left Noetherian and A/N is left Artinian.*

4) *A is almost left Noetherian and π-regular, and A/N is left s-unital.*

5) *A is almost left Noetherian and every proper prime ideal of A is an almost maximal left ideal.*

6) *N is nilpotent, ${}_A(AN^{i-1}/N^i)$ is finitely generated for all $i>0$, A satisfies the ascending chain condition for semiprime ideals, and every proper prime ideal of A is an almost maximal left ideal.*

Proof. 1) $\Longleftrightarrow$ 2) $\Longleftrightarrow$ 3). Under any of the conditions 1) — 3), N is nilpotent: $N^n=0$, and ${}_A A/N$ is completely reducible (Theorem 3 (2) and Proposition 3). Observe the descending chain $A\supseteq N\supseteq N^2\supseteq\cdots\supseteq N^n=0$. By Theorem 1 (1), ${}_A(N^{i-1}/N^i)$ is almost Artinian if and only if ${}_A(AN^{i-1}/N^i)$ is Artinian, or equivalently ${}_A(N^{i-1}/N^i)$ is almost Noetherian. Hence, by Proposition 2 all the conditions 1) — 3) are equivalent.

1) $\Longrightarrow$ 4) $\Longrightarrow$ 3). By Propositions 2 (2), 3 and Lemma 4.

5) $\Longrightarrow$ 6). By Theorem 3 (1), A satisfies the ascending chain condition for semiprime ideals. Hence, by [4, Theorem 3], $N=\bigcap_{i=1}^{k} P_i$ with

some prime ideals P_i. Since ${}_A(\cap_{i=1}^{j-1} P_i)/(\cap_{i=1}^{j} P_i) \simeq {}_A(P_j + \cap_{i=1}^{j-1} P_i)/P_j$ and ${}_A A/P_j$ is completely reducible, we see that ${}_A A/N$ is Artinian (Lemma 1). Now, 6) is obvious by Theorem 1 (1) and Theorem 3 (2).

6) $\Longrightarrow$ 1). Again by [4, Theorem 3], $N = \cap_{i=1}^{k} P_i$ with some prime ideals P_i, and ${}_A A/P_i$ $(\simeq {}_A(A/N)/(P_i/N))$ is a completely reducible module of finite length. Hence, A/N is Artinian semisimple. Then, by Theorem 1 (2), ${}_A N$ is almost Artinian, and therefore A is almost left Artinian by Proposition 2 (1).

The next is an easy combination of [3, Theorem 9] and Theorem 4.

Corollary 2. *If A is almost left Artinian then the full matrix ring $(A)_m$ is almost left Artinian, and eAe is left Artinian for every idempotent e of A.*

Corollary 3 (cf. [1, Theorems 3 and 12]). (1) *A (left and right) duo ring A is almost left Artinian if and only if A is the direct sum of an Artinian ring with identity and a nilpotent ring.*

(2) *A left duo ring A is almost left Artinian if and only if A is almost left Noetherian and every proper prime ideal of A is maximal.*

Proof. (2) is immediate from Theorem 4. It remains only to prove the only if part of (1). Let e be an idempotent lifted from the identity of A/N (Proposition 3). Since A is a duo ring, Ae coincides with eA. Hence A is the direct sum of the Artinian ring $eAe = Ae$ (Corollary 2) and the nilpotent ideal $l(e)$ contained in N (Proposition 3).

Remark. Let A be an almost left Artinian ring. If $AN = N$ then, as was claimed in [1, p. 17], A is left Noetherian by Proposition 3 and Theorem 1 (1). However, this is a consequence of Hopkins' theorem, too. In fact, by [7, Theorem 1], A has then a left identity.

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