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COMMUTATIVITY THEOREMS FOR s-UNITAL RINGS SATISFYING POLYNOMIAL IDENTITIES

Dedicated to Professor N. Jacobson on his 70th birthday

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Recently, H. E. Bell [2] proved that an n-torsion free ring with identity which satisfies the identity $(xy)^n=(yx)^n$ is necessarily commutative. More recently, Y. Hirano, M. Hongan and the second author of this paper has proved the same for s-unital rings [5]. On the other hand, the first author of this paper has proved that an $(n+1)n$-torsion free ring with identity which satisfies the identity $(xy)^{n+1}=x^{n+1}y^{n+1}$ is commutative [1]. Our objective is to generalize Bell's result to s-unital rings satisfying weaker identities which are implied by the identity $(xy)^n=(yx)^n$, and to generalize the main theorem of [1] to s-unital rings. Following [5], a ring R is called *s-unital* if for each x in R, $x\in Rx\cap xR$. As stated in [5], if R is an s-unital ring, then for any finite subset F of R, there exists an element e in R such that $ex=xe=x$ for all x in F. Such an element e will be called a *pseudo-identity* of F.

Throughout, R will represent a ring with center C, and N will denote the set of all nilpotent elements of R. As usual, $[x, y]$ will denote the commutator $xy-yx$. Our present objective is to prove the following theorems.

Theorem 1. *Let n be a fixed positive integer. Let R be an s-unital ring in which every commutator is n-torsion free. If R satisfies the polynomial identities $[x^n, y^n]=0$ and $[x, (xy)^n-(yx)^n]=0$, then R is commutative.*

Theorem 2. *Let n be a fixed positive integer. Let R be an s-unital ring in which every commutator is $(n+1)n$-torsion free. If R satisfies the polynomial identity $(xy)^{n+1}-x^{n+1}y^{n+1}=0$, then R is commutative.*

In preparation for the proof of our theorems, we establish the following lemmas.

Lemma 1. *Let m, n be fixed positive integers.*

(1) *If $[a,[a,b]]=0$ then $[a^n,b]=na^{n-1}[a,b]$, where a, $b\in R$.*

(2) *Let e be a pseudo-identity of $\{a,b\}\subseteq R$. If $a^m b=0=(a+e)^m b$ then $b=0$.*

(3) *If R satisfies the polynomial identity $[x^n,y^m]=0$ or $(xy)^{n+1}-x^{n+1}y^{n+1}=0$, then the commutator ideal $D(R)$ of R is contained in N.*

(4) *If R is an s-unital ring satisfying the polynomial identity $[x^n,y^n]=0$, then there exists a positive integer k such that $k[x^n,y]=0$.*

Proof. (1) is well known. (3) is obvious by [4, Theorem] and [3, Theorem 1], and (4) is proved in [5, Lemma 10].

(2) We have

$0=a^{m-1}(a+e)^m b=a^{m-1}b$, and

$0=(-1)^m(a+e)^{m-1}a^m b=(-1)^m(a+e)^{m-1}\{-e+(a+e)\}^m b=(a+e)^{m-1}b$.

Continuing this process, we obtain eventually $b=0$.

Lemma 2. *Let n be a fixed positive integer. Let R be an s-unital ring in which every commutator is n-torsion free.*

(1) *If R satisfies the polynomial identity $nx^m[x,y]=0$ with a non-negative integer m, then R is commutative.*

(2) *If R satisfies the polynomial identity $[x^n,y]=0$, then R is commutative.*

Proof. (1) Let a, b be arbitrary elements of R, and e a pseudo-identity of $\{a,b\}$. Since $na^m[a,b]=0$ and $n(a+e)^m[a,b]=0$, by Lemma 1 (2) we have $n[a,b]=0$, and therefore $[a,b]=0$.

(2) Since $D(R)\subseteq N$ by Lemma 1 (3), from the proof of [5, Lemma 9] it follows that $N\subseteq C$. Hence, by Lemma 1 (1), $nx^{n-1}[x,y]=[x^n,y]=0$. Now, the commutativity of R is obvious by (1).

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. First, we shall show that $[u,d^n]=0$ for every $d\in R$ and every $u\in N$. Let f be a pseudo-identity of $\{d,u\}$. Since u is nilpotent, there exists a minimal positive integer m such that $[u^i,d^n]=0$ for all integers $i\geq m$. If $m\geq 2$, then

$0=[(f+u^{m-1})^n,d^n]=[f^n+nu^{m-1}+\cdots+u^{(m-1)n},d^n]=n[u^{m-1},d^n]$, and hence $[u^{m-1},d^n]=0$, which contradicts the minimality of m. Thus, $m=1$, and $[u,d^n]=0$.

According to Lemma 1 (4), there exists a positive integer k such that $k[x^n,y]=0$. Since $D(R)\subseteq N$ by Lemma 1 (3), it follows from what was just shown above that $[x^n,[x^n,y]]=0$. Hence, by Lemma 1 (1), $[x^{nk},y]$

$=kx^{n(k-1)}[x^n, y]=0$. Now, let a, b be arbitrary elements of R, and let e be a pseudo-identity of $\{a, b\}$. Then, combining the above with the second polynomial identity, we have

$$0=[a, (a^{nk}b)^n-(a^{nk-1}ba)^n]=[a, a^{n^2k}b^n-a^{n^2k-1}b^n a]=a^{n^2k-1}[a, [a, b^n]].$$

Similarly, we have $0=(a+e)^{n^2k-1}[a, [a, b^n]]$. We obtain therefore $[a, [a, b^n]]=0$ (Lemma 1 (2)) and $na^{n-1}[a, b^n]=0$ (Lemma 1 (1)). Again by Lemma 1 (2), $n[a, b^n]=0$, and hence $[a, b^n]=0$. Now, R is commutative by Lemma 2 (2).

From the proof of [5, Theorem 3], one will easily see that if R is an s-unital ring in which every commutator is n-torsion free then the polynomial identity $(xy)^n-(yx)^n=0$ implies $[x^n, y^n]=0$. Hence, Theorem 1 implies the following:

Corollary 1. *Let R be an s-unital ring in which every commutator is n-torsion free. If R satisfies the identity $(xy)^n=(yx)^n$ then R is commutative.*

Proof of Theorem 2. First, we shall show that $[u, d^{n+1}]=0$ for every $d\in R$ and every $u\in N$. Let f be a pseudo-identity of $\{d, u\}$. If u_0 is the quasi-inverse of u, then $fu_0=u_0f=u_0$ and the map $\sigma: R\to R$ defined by $x\to x-u_0x-xu+u_0xu$ is a ring automorphism of R. By hypothesis,

$$\begin{aligned}0&=(f-u)^{n+1}\{(f-u_0)^{n+1}d^{n+1}(f-u)^{n+1}\}(f-u_0)-d^{n+1}(f-u)^n\\&=(f-u)^{n+1}\sigma(d)^{n+1}(f-u_0)-d^{n+1}(f-u)^n\\&=(f-u)^{n+1}\sigma(d^{n+1})(f-u_0)-d^{n+1}(f-u)^n\\&=(f-u)^n d^{n+1}-d^{n+1}(f-u)^n=[(f-u)^n, d^{n+1}].\end{aligned}$$

Choose the minimal positive integer m such that $[u^i, d^{n+1}]=0$ for all $i\geq m$. Suppose $m>1$. Then, by the above, $[(f-u^{m-1})^n, d^{n+1}]=0$. Combining this with $[u^i, d^{n+1}]=0$ $(i\geq m)$, we get $n[u^{m-1}, d^{n+1}]=0$, and hence $[u^{m-1}, d^{n+1}]=0$. But this contradicts the minimality of m. Thus, $k=1$, and hence $[u, d^{n+1}]=0$.

Let R^* be the subring generated by all $(n+1)$-th powers of elements of R. Then, it follows from what was just shown above that the set N^* of nilpotent elements of R^* is contained in the center C^* of R^*. Moreover, by Lemma 1 (3), $D(R^*)\subseteq N^*\subseteq C^*$. Let a^*, b^* be arbitrary elements of R^*. Then both $[a^*, b^{*n+1}]$ and $[a^{*n}, b^{*n+1}]$ are in C^*. Hence, by Lemma 1 (1),

$$
\begin{aligned}
na^{*n+1}[a^*, b^{*n+1}] &= a^{*2}[a^{*n}, b^{*n+1}] = a^*[a^{*n}, b^{*n+1}]a^* \\
&= a^{*n+1}b^{*n+1}a^* - a^*b^{*n+1}a^{*n+1} \\
&= (a^*b^*)^{n+1}a^* - a^*(b^*a^*)^{n+1} = 0.
\end{aligned}
$$

According to Lemma 1 (2), it follows then that $n[a^*, b^{*n+1}]=0$, and therefore $[a^*, b^{*n+1}]=0$. Now, by Lemma 2 (2), $[a^*, b^*]=0$, and hence for all x, y in R, $[x^{n+1}, y^{n+1}]=0$. Combining this with the polynomial identity $(xy)^{n+1}-x^{n+1}y^{n+1}=0$, we obtain $(xy)^{n+1}=(yx)^{n+1}$. Hence, R is commutative by Corollary 1.

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