Math. J. Okayama Univ. 22 (1980), 107—109

A COMMUTATIVITY THEOREM FOR RINGS WITH CONSTRAINTS INVOLVING AN ADDITIVE SUBSEMIGROUP. II

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Throughout the present paper, R will represent an associative ring with center Z, N the set of all nilpotent elements of R, and A a subset of R such that $aa'=a'a$ and $a+a'\in A$ for all $a, a'\in A$. Given $x, y\in R$, we set $[x, y]=xy-yx$.

The purpose of this paper is to prove the following commutativity theorem.

Theorem 1. *Suppose that* (i) *for every x in R, there exists a positive integer $m=m(x)$ such that $x-x^{m+1}\in A$,* (ii) *$[x, [x, a]]=0$ for all $a\in A$ and $x\in R$,* (iii) *$x-y\in A$ implies $x^n=y^n$ with some positive integer $n=n(x, y)$ or both $x, y\in V_R(A)$, the centralizer of A in R. Then R is commutative.*

Our theorem recovers a well known theorem of Herstein, which asserts that a ring which satisfies condition (i) above with $A=Z$ is commutative [2, p. 221].

In preparation for the proof of our theorem, we establish the following lemmas.

Lemma 1. (a) *If* (i) *is satisfied, then $N\subseteq A$.*

(b) *If* (iii) *is satisfied, then for each $x\in R$ and $a\in A$ there exists a positive integer n such that $[x^n, a]=0$; in particular, every idempotent of R is in $V_R(A)$.*

(c) *If* (i), (ii), (iii) *are satisfied, then any subring of R and any homomorphic image of R inherit all of these hypotheses.*

Proof. (a) Suppose $x^k=0$. By (i), there exist $n_1, \cdots, n_k$ with each $n_i>1$ such that $x-x^{n_1}\in A$, $x^{n_1}-x^{n_1n_2}\in A, \cdots, x^{n_1\cdots n_{k-1}}-x^{n_1\cdots n_k}\in A$. Hence, $x=x-x^{n_1\cdots n_k}=(x-x^{n_1})+(x^{n_1}-x^{n_1n_2})+\cdots+(x^{n_1\cdots n_{k-1}}-x^{n_1\cdots n_k})\in A$.

The proofs of (b) and (c) are quite similar to those of [3, Lemma 1 (b)] and [3, Corollary 1], respectively.

Lemma 2. *Suppose* (i) *and* (iii) *are satisfied. Then we have:*

(a) *Every idempotent of R is central.*

(b) *N is an ideal of R.*

(c) *Given an element x of R not contained in $V_R(A)$, there exist positive integers m, n such that $x^n = x^{n(m+1)}$.*

Proof. In view of Lemma 1 (a), (b), the proof of the first two statements proceeds in the same way as that of [1, Lemma 2]. (Only minor modifications are needed in the proof of (b).)

(c) By (i), $x - x^{m+1} \in A$ with some positive integer m. Then, since $x \notin V_R(A)$, there exists a positive integer n such that $x^n = x^{n(m+1)}$, by (iii).

We are now in a position to prove the main theorem of this paper.

Proof of Theorem 1. As is well known, R is a subdirect sum of subdirectly irreducible rings. Hence, in view of Lemma 1 (c), we may assume that R is subdirectly irreducible. According to Herstein's theorem [2, p. 221], it is enough to prove $A \subseteq Z$. Suppose on the contrary that there exist $a \in A$ and $x \in R$ such that $ax \neq xa$. Then, by Lemma 1 (a) and Lemma 2 (c), x is not nilpotent and $x^n = x^{n(m+1)}$ with some positive integers m, n. Since x^{nm} is a non-zero central idempotent (Lemma 2 (a)) and R is subdirectly irreducible, x^{nm} must be 1. We claim that $a(kx) \neq (kx)a$ for some integer $k > 1$. In fact, a cannot commute with both $2x$ and $3x$. Then, repeating the above argument for kx, we have $(kx)^{n'm'} = 1$ with some positive integers m', n'. Then we readily obtain $k^{nmn'm'} - 1 = 0$, which implies that the characteristic of the subdirectly irreducible ring R is p^{α} with a prime p. Since N forms an ideal by Lemma 2 (b), $\bar{R} = R/N$ is a reduced ring. Let $\bar{x} \in \bar{R}$. Evidently, the finite reduced ring $\langle \bar{x} \rangle$ is a direct sum of finite fields of characteristic p. We can find therefore an integer $\beta \geq \alpha$ such that $\bar{x} = \bar{x}^{p^{\beta}}$, and hence $x - x^{p^{\beta}} \in N \subseteq A$ (Lemma 1 (a)). Since $[x, [x, a]] = 0$ by (ii), an easy induction proves $[x^{p^{\beta}}, a] = p^{\beta} x^{p^{\beta}-1}[x, a] = 0$. Combining this with $[x - x^{p^{\beta}}, a] = 0$, we have a contradiction $[x, a] = 0$. This contradiction proves the theorem.

Remarks. (a) Let A' be the additive subgroup generated by A. Then, the hypotheses (ii) and (iii) are equivalent to those concerning A' (instead of A).

(b) In Theorem 1, the hypothesis that A is commutative is essential. To see this, let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in \mathrm{GF}(3) \right\}$, and $A = N$. It can be checked that all the hypotheses (i) (for $m = 2$), (ii), (iii) (for $n = 3$) are satisfied. Note, however, that A is not commutative.

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(*Received December 11, 1979*)