

ON THE BROWDER-LIVESAY INVARIANT OF FREE INVOLUTIONS ON HOMOLOGY 3-SPHERES

Dedicated to Professor A. Komatu on his 70th birthday

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Introduction. Let M be a 3-dimensional homology sphere. Let T be a free involution on M . Let $\alpha(M, T)$ be the Browder-Livesay invariant, and let $\mu(M)$ be the μ -invariant of M . Then $\alpha(M, T) \in 8\mathbb{Z}$ and $\mu(M) \in 8\mathbb{Z} \pmod{16}$. In §1, we shall prove that $\mu(M) = \alpha(M, T) \pmod{16}$ (Theorem 1). This equation has been proved for Seifert homology spheres by W. D. Neumann and R. Raymond [11]. By this equation, in order to compute the μ invariant of those homology spheres which have free involution, it suffices to compute their Browder-Livesay invariant. For this purpose, in §2, we shall define 'signature invariants' $\sigma_{(m,n)}(V, T)$ for orientation preserving free involutions on 3-dimensional homology circles and coprime integer pairs (m, n) with m odd. In §3, by making use of these invariants, we shall give a formula of the Browder-Livesay invariant of some free involutions on homology spheres (Theorem 3 and its Corollary). This formula is a generalization of the formula given by W. D. Neumann [10] for Seifert homology spheres.

All the homology and cohomology groups in this paper have integer coefficients unless otherwise mentioned.

1. Browder-Livesay invariant and μ -invariant. For convenience, we shall work in the smooth category. All the manifolds will be compact and oriented. The boundary of X , ∂X , inherits its orientation from X . $-X$ will denote X with the opposite orientation. The unit n -disc will be denoted by D^n , the $(n-1)$ -sphere, ∂D^n , by S^{n-1} , and the unit interval by I .

If X, Y are disjoint n -manifolds with X_0, Y_0 $(n-1)$ submanifolds of $\partial X, \partial Y$ respectively, and if $h: X_0 \rightarrow Y_0$ is an orientation reversing diffeomorphism, then $X \cup_h Y$ will denote the quotient space $X \cup Y / (x \sim hx \text{ for all } x \in X_0)$. By the existence and uniqueness of collars, $X \cup_h Y$ has a natural structure as an n -manifold. If h comes from some natural identifications of X_0 and Y_0 with Z , say, we may write $X \cup_Z Y$ for $X \cup_h Y$.

A homology sphere (resp. \mathbb{Z}_2 homology sphere) is a closed 3-manifold M such that $H_*(M) \cong H_*(S^3)$ (resp. $H_*(M, \mathbb{Z}_2) \cong H_*(S^3, \mathbb{Z}_2)$). For a \mathbb{Z}_2 homology sphere M , Eells and Kuiper [2] have defined an invariant $\mu(M) \in 2\mathbb{Z} \pmod{16}$ as follows: M bounds a 4-manifold Y such that $H_1(Y)$

has no 2-torsion and the quadratic form of Y is even. Then, we set

$$\mu(M) = -\sigma(Y) \pmod{16},$$

where $\sigma(Y)$ is the signature of Y . This is well-defined, by Rohlin's theorem [12]. If M is a homology sphere, then the intersection form of Y is unimodular, and hence $\sigma(Y) \equiv 0 \pmod{8}$, and the possible value of $\mu(M)$ is 8 or 0.

Let T be a free involution on a \mathbb{Z}_2 homology sphere M . Then T preserves the orientation of M by Lefschetz fixed point theorem. The Browder-Livesay invariant $\alpha(M, T)$ is defined as follows ([1]): There is a decomposition of M such that $M = A \cup TA$, $A \cap TA = B$, where A is a 3-dimensional submanifold of M with $\partial A = B$, and B is an invariant closed surface in M (B is called a characteristic surface of (M, T)). A inherits its orientation from M . Put $K = \text{Ker } (H_1(B) \rightarrow H_1(A))$. Let $\langle \cdot, \cdot \rangle$ be the bilinear form on K defined by $\langle x, y \rangle = x \cdot T_* y$ for $x, y \in K$, where T_* is the homomorphism $H_1(B) \rightarrow H_1(B)$ induced by T , and dot denotes the intersection number. Then $\langle \cdot, \cdot \rangle$ is a non-degenerate symmetric bilinear form on K of even type. Set

$$\alpha(M, T) = \sigma(\langle \cdot, \cdot \rangle),$$

where σ is the signature of $\langle \cdot, \cdot \rangle$. If M is a homology sphere, the bilinear form $\langle \cdot, \cdot \rangle$ is unimodular, and hence $\alpha(M, T) \equiv 0 \pmod{8}$.

The rest of this section will be devoted to the proof of the following

Theorem 1. *Let T be a free involution on a homology sphere M . Then $\mu(M) = \alpha(M, T) \pmod{16}$.*

Proof. Let M/T be the orbit space of M and let $p: M \rightarrow M/T$ be the covering projection. Let \bar{C} be a simple closed curve in M/T which represents the generator of $H_1(M/T, \mathbb{Z}_2) = \mathbb{Z}_2$. Then $C = p^{-1}(\bar{C})$ is an invariant simple closed curve in M . Let U be an invariant closed tubular neighborhood of C in M . Then by Alexander duality theorem, it follows that $\overline{M-U}$ is a homology circle, $H_*(\overline{M-U}) = H_*(S^1)$, where $\overline{M-U}$ is the closure of $M-U$ in M . Now, both ∂U and $p(\partial U)$ are $S^1 \times S^1$. Let \bar{D} be a simple closed curve in $p(\partial U)$ which is homologous to zero in $p(\overline{M-U})$. Then $p^{-1}(\bar{D})$ consists of two simple closed curves, say, D and its transformed image TD . Now there is an orientation preserving equivariant embedding $f: S^1 \times D^2 \rightarrow M$ such that $f(S^1 \times D^2) = U$, $f(S^1 \times 0) = C$ and $f(S^1 \times 1) = D (\subset \partial \overline{M-U} = \partial U)$, where $S^1 \times D^2$ has the involution T defined by $T(x, y) = (-x, -y)$ for $(x, y) \in S^1 \times D^2$, and D^2 is regarded as the unit

disc in the complex plane. Let $D^2 \times D^2$ be the 4-disc with the involution T defined by $T(x, y) = (-x, y)$ ($(x, y) \in D^2 \times D^2$). Let $g: \partial D^2 \times D^2 \rightarrow M$ be the equivariant embedding defined by $g(x, y) = f(x, xy)$ for $(x, y) \in \partial D^2 \times D^2$. Form the manifold $M \times I \cup_g D^2 \times D^2$, where g is considered as an embedding of $\partial D^2 \times D^2$ to $M \times I$. Let M_0 be the manifold $\overline{M - U} \cup_{g|_{\partial D^2 \times \partial D^2}} D^2 \times \partial D^2$, where $|$ denotes the restriction of the map. Then $\partial(M \times I \cup_g D^2 \times D^2) = M \cup -M_0$. By Mayer-Vietoris theorem, it follows that M_0 is a homology sphere. Now M_0 has an involution T with fixed point set $C_0 = 0 \times \partial D^2$. Let $M_1 = M_0/T$ be the orbit space, and let $p_0: M_0 \rightarrow M_1$ be the projection. Then it can be seen that any closed curve in M_1 intersecting transversely and non-vacuously with $C_1 = p_0(C_0)$ lifts to a closed curve in M_0 . Hence $p_{0*}: \pi_1(M_0) \rightarrow \pi_1(M_1)$ is onto. This implies that M_1 is a homology sphere. M_0 is considered as the 2-fold branched covering space of M_1 branched over the curve C_1 . Let W be a 4-dimensional spin manifold which bounds M_1 , $\partial W = M_1$, and is simply connected. Now $K_1 = (M_1, C_1)$ is a knot in the homology sphere M_1 . There is a Seifert surface of K_1 , say S , in M_1 with $\partial S = C_1$ ([4]). Since S has a trivial normal bundle in M_1 , there is an embedding of $S \times [-1, 1]$ into M_1 such that $S \times 0 = S$. Now take two copies of W , say W_1 and W_2 , and form the manifold $X = W_1 \cup_{S \times [-1, 1]} W_2$ from the disjoint union $W_1 \cup W_2$ by identifying the point $(x, t) \in S \times [-1, 1] \subset \partial W_1$ with the point $(x, -t) \in S \times [-1, 1] \subset \partial W_2$. Then X has a natural involution T which is orientation preserving and interchanges W_1 with W_2 and W_2 with W_1 . The fixed point set of T is $S = S \times 0$ and $\partial X \cap S = \partial S$. The orbit space X/T is W . ∂X is the 2-fold branched covering space of M_1 branched over the curve $\partial S = C_1$, and ∂X is equivariantly diffeomorphic to M_1 . Let Y be the manifold $(M \times I \cup_g D^2 \times D^2) \cup_{M_0} X$ obtained from the disjoint union $(M \times I \cup_g D^2 \times D^2) \cup X$ by identifying the common boundary M_0 . Then $\partial Y = M$ and Y has an orientation preserving involution T which restricts to the original involution on M . Now we compute $\alpha(M, T)$ and $\mu(M)$ by making use of Y .

First we consider the homology of $X = W_1 \cup_{S \times I} W_2$. By Mayer-Vietoris theorem, there is the following exact sequence:

$$0 \rightarrow H_2(W_1) \oplus H_2(W_2) \rightarrow H_2(X) \xrightarrow{\partial_*} H_1(S \times I) \rightarrow 0.$$

Let $\{\alpha_j\}_{j=1, \dots, s}$ be a base of $H_2(W)$ and let $\{\alpha_j^{(i)}\}$ be the corresponding base of $H_2(W_i)$ ($i=1, 2$), where $s = \text{rank } H_2(W)$. Then $T_* \alpha_j^{(1)} = \alpha_j^{(2)}$ in $H_2(X)$, where T_* is the homomorphism induced by the involution T on X ($j=1, \dots, s$). Let $\{E_k\}_{k=1, \dots, t}$ be curves in S which represent a base of $H_1(S)$, where $t = \text{rank } H_1(S)$. Since each E_k is homologous to zero in

$M_1 = \partial W$, there exists a 2-chain G_k in a collar neighborhood $\partial W \times I$ such that $\partial G_k = E_k$. Let $G_k^{(i)}$ be the copies of G_k in W_i ($i=1, 2$). Then $TG_k^{(1)} = G_k^{(2)}$ and $TG_k^{(2)} = G_k^{(1)}$. Now the chain $G_k^{(1)} - G_k^{(2)}$ is a cycle and represents a homology class β_k in $H_2(X)$ such that $\partial_* \beta_k = [E_k] \in H_1(S)$, where $[E_k]$ is the homology class represented by the curve E_k ($k=1, \dots, t$). Clearly $T_* \beta_k = -\beta_k$ in $H_2(X)$. By the above Mayer-Vietors tequence, $\{\alpha_j^{(1)}, \alpha_j^{(2)}, \beta_k\}$ ($j=1, \dots, s$ and $k=1, \dots, t$) forms a base of $H_2(X)$. Next we consider $H_2(M \times I \cup_{c,1} D^2 \times D^2)$. Let ξ be the homology class represented by the chain $C \times I \cup_{c,1} (0 \times D^2)$, $\xi \in H_2(M \times I \cup_{c,1} D^2 \times D^2, \partial) = H_2(M \times I \cup_{c,1} D^2 \times D^2)$. Then $\xi \cdot \xi = 1$ by the construction and ξ generates $H_2(M \times I \cup_{c,1} D^2 \times D^2)$, and $T_* \xi = \xi$.

Now the set $\{\xi, \alpha_j^{(1)}, \alpha_j^{(2)}, \beta_k\}$ ($j=1, \dots, s$ and $k=1, \dots, t$) is a base of $H_2(Y)$ such that

- (1) $T_* \xi = \xi$, $T_* \alpha_j^{(1)} = \alpha_j^{(2)}$ and $T_* \beta_k = -\beta_k$,
- (2) $\xi \cdot \xi = 1$ and $\xi \cdot \alpha_j^{(1)} = \xi \cdot \alpha_j^{(2)} = \beta_k \cdot \alpha_j^{(1)} = \beta_k \cdot \alpha_j^{(2)} = \alpha_j^{(1)} \cdot \alpha_j^{(2)} = 0$,

where dot denotes the intersection number ((2) is clear from the construction of ξ and β_k). Let $A^{(i)}$ be the intersection matrix $(\alpha_k^{(i)} \cdot \alpha_m^{(i)})$ ($i=1, 2$ and $k, m=1, \dots, s$) and let B be the intersection matrix $(\beta_k \cdot \beta_m)$ ($k, m=1, \dots, t$).

Assertion (1). $\alpha(M, T)^* = -\text{sign } B$.

Proof. The fixed point set of the involution T on Y is the closed 2-surface $F = 0 \times D^2 \cup_{c,0} S$. By Hirzebruch [5],

$$\alpha(M, T) = \text{sign}(Y, T) - [F] \cdot [F],$$

where $\text{sign}(Y, T)$ denotes the signature of the bilinear form $\langle \ , \ \rangle$ on $H_2(Y)$ defined by $\langle x, y \rangle = x \cdot T_* y$ for $x, y \in H_2(Y)$, and $[F]$ is the homology class represented by F . With respect to the above base of $H_2(Y)$, the bilinear form $\langle \ , \ \rangle$ is represented by the matrix

$$\langle 1 \rangle + \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix} + \langle -B \rangle,$$

where $A = A^{(1)} = A^{(2)}$. Hence $\text{sign}(Y, T) = 1 - \text{sign } B$. Now $[F] \cdot [F] = 1$, and the Assertion (1) follows.

Assertion (2). $\mu(M) = -\text{sign } B \pmod{16}$.

Proof. With respect to the above base, the intersection form on $H_2(Y)$ is represented by that matrix

$$\langle 1 \rangle + A + A + B,$$

where $A = A^{(1)} = A^{(2)}$. Hence $\sigma(Y) = 1 + 2 \operatorname{sign} A + \operatorname{sign} B$. Since $\partial W = M_1$ is a homology sphere and W is a spin manifold, A is a unimodular symmetric matrix with even diagonal entries. Therefore $\operatorname{sign} A \equiv 0 \pmod{8}$, and we obtain $\sigma(Y) = 1 + \operatorname{sign} B \pmod{16}$. Now the class ξ is represented (as a relative cycle) by a disc $C \times I \cup_{C \times 1} (0 \times D^3)$ with boundary $C \subset M$, and ξ is characteristic for the quadratic form of Y , that is, $\xi \cdot z = z \cdot z \pmod{2}$ for every $z \in H_2(Y)$. Let K be the knot (M, C) in the homology sphere M . Then Gordon [4] has shown that $\mu(M) = \xi \cdot \xi - \sigma(Y) - c(K) \pmod{16}$, where $c(K)$ is the Arf invariant of the knot $K = (M, C)$. Hence $\mu(M) = -\operatorname{sign} B - c(K) \pmod{16}$. Now it suffices to prove that $c(K) = 0 \pmod{16}$. Let $J_K(t)$ be the Alexander polynomial of K . Then $c(K) = (J_K(-1))^2 - 1 \pmod{16}$ ([7]). Now the knot $K = (M, C)$ and the knot $K_0 = (M_0, C_0)$ have the same knot complement, so that $J_K(t) = J_{K_0}(t)$. Since M_1 is the 2 fold branched covering space of M_0 branched over the curve C_1 , $J_{K_0}(t^2) = J_{K_1}(t) J_{K_1}(t^{-1})$, where $K_1 = (M_1, C_1)$ is the knot in the homology sphere M_1 ([3]). Hence $J_K(-1) = J_{K_0}(-1) = J_{K_1}(\sqrt{-1}) J_{K_1}(-\sqrt{-1})$. Since the Alexander polynomial of any knot is a reciprocal polynomial, it can be seen that $J_{K_1}(\sqrt{-1}) J_{K_1}(-\sqrt{-1})$ is a square of an odd integer, say, $(2d+1)^2$ (note that $J_{K_1}(-1) \equiv 1 \pmod{2}$). Therefore $c(K) = (J_K(-1))^2 - 1 = 8d(d+1) \pmod{16}$. This proves Assertion (2), and therefore Theorem 1.

Remark 1. Let $K_1 = (M_1, C_1)$ be the knot in the above proof. Then it is well known that $\operatorname{sign} B$ in the above proof is equal to the so-called signature of the knot K_1 , $\sigma(K_1)$. Hence we have proved that $\alpha(M, T) = -\sigma(K_1)$.

Remark 2. The equation in Theorem 1 has been proved for Seifert homology spheres by W. D. Neumann and F. Raymond [11]. Many examples of free involutions on homology spheres which cannot be embedded in S^1 action have been constructed by C. McA. Gordon [4]. The same argument in the above proof gives only $\mu(M) = \alpha(M, T) \pmod{4}$ for Z_2 homology spheres with free involution. In general, the equality $\mu(M) = \alpha(M, T) \pmod{16}$ does not hold for Z_2 homology spheres with free involution. As an example, let $L(3, 1)$ be the standard Lens space and let M be the connected sum $L(3, 1) \# L(3, 1)$. Then M has a natural free involution with S^2 as the characteristic surface. Hence $\alpha(M, T) = 0$. On the other hand, $\mu(M) = 2\mu(L(3, 1)) = 4 \pmod{16}$.

2. Signatures of orientation preserving involutions on homology

circles. Let V be a homology circle, that is, a compact 3-dimensional manifold with boundary $S^1 \times S^1$ such that $H_*(V) = H_*(S^1)$. Let T be an orientation preserving free involution on V . Let V/T be the orbit space and let $p: V \rightarrow V/T$ be the covering projection.

Lemma 2.1. V/T is a homology circle.

Proof. Let S^∞ be the infinite dimensional sphere with the antipodal free involution T . Then $S^\infty/T = P^\infty$ is the infinite dimensional real projective space and $S^\infty \rightarrow P^\infty$ is the universal Z_2 bundle. Let $S^\infty \times_T V$ be the orbit space of $S^\infty \times V$ by the diagonal action of Z_2 . Since the involution on V is free and S^∞ is contractible, $S^\infty \times_T V$ is homotopy equivalent to V/T . The projection to the first factor $S^\infty \times_T V \rightarrow P^\infty$ is a fibre bundle with fibre V . Considering the cohomology Serre spectral sequence of this fibre bundle, it follows that $p^*: H^*(V/T) \rightarrow H^*(V)$ is injective and $p^*(H^1(V/T)) = 2\mathbb{Z} \subset \mathbb{Z} = H^1(V)$. This proves Lemma 2.1.

Definition 2.1. A pair of characteristic curves in ∂V is defined as a pair of oriented simple closed curves (C, D) in V such that (i) $TC = C$ and $D \cap TD = \emptyset$, and (ii) $C \cap D$ consists of a single point and $C \cdot D = 1$, where dot denotes the intersection number.

Lemma 2.2. For $i = 1, 2$, let V_i be a homology circle with free orientation preserving involution T_i . Let (C_i, D_i) be a pair of characteristic curves in ∂V_i . Then there is an orientation reversing equivariant diffeomorphism $h: \partial V_1 \rightarrow \partial V_2$ such that h maps C_1 onto C_2 orientation preservingly and D_1 onto D_2 orientation reversingly.

Proof. Let $p_i: V_i \rightarrow V_i/T_i$ be the covering projection ($i = 1, 2$). Then $(p_i(C_i), p_i(D_i))$ is a pair of simple closed curves in $\partial(V_i/T_i)$ whose homology classes generate $H_1(\partial(V_i/T_i))$. Since $\partial(V_i/T_i)$ is $S^1 \times S^1$, there is a diffeomorphism $\bar{h}: \partial(V_1/T_1) \rightarrow \partial(V_2/T_2)$ such that \bar{h} maps $p_1(C_1)$ onto $p_2(D_2)$ orientation preservingly and $p_1(D_1)$ onto $p_2(C_2)$ orientation reversingly. Now it suffices to take as h the map which covers \bar{h} .

Definition 2.2. A pair of characteristic curves in ∂V , (C, D) , is called of type (m, n) if $[C] = m\lambda$ and $[D] = n\lambda$ in $H_1(V)$, where λ is a base of $H_1(V) = \mathbb{Z}$ and $[C]$ and $[D]$ are represented classes. We always choose λ so that $m > 0$.

The integer pairs (m, n) in the above definition are coprime pairs if $n \neq 0$.

Lemma 2.3. *Let (C, D) be a pair of characteristic curves in ∂V of type (m, n) . Then m is an odd integer.*

Proof. Let $p: V \rightarrow V/T$ be the covering projection. Then $p(C)$ is a simple closed curve in $\partial(V/T)$. Let $\bar{\lambda}$ be the base of $H_1(V/T)$ such that $p_*(\lambda) = 2\bar{\lambda}$, where p_* is the induced homomorphism. Then $[p(C)] = m\bar{\lambda}$ in $H_1(V/T)$. The map p is a covering map associated to the composite homomorphism $\rho: \pi_1(V/T) \rightarrow H_1(V/T) = \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$, where the last map is the mod 2 reduction. Since $C \rightarrow p(C)$ is a non-trivial covering, $\rho(p(C)) \equiv m \not\equiv 0 \pmod{2}$. Hence m is odd.

First we consider a pair of characteristic curves in ∂V of type $(1, 0)$, (C, D) . The curve D is null-homologous to zero in V .

Lemma 2.4. *Let (C, D) be a pair of characteristic curves in ∂V of type $(1, 0)$. Then there is a connected surface B in V such that $B \cap TB = \emptyset$, $\partial B = B \cap \partial V = D$ and $H_1(B) \rightarrow H_1(V)$ is a zero-homomorphism, where the last map is induced by the inclusion.*

Proof. In the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(V) & \xrightarrow{p_*} & \pi_1(V/T) & \longrightarrow & \mathbb{Z}_2 \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & H_1(V) & \xrightarrow{p_*} & H_1(V/T) & \longrightarrow & \mathbb{Z}_2 \longrightarrow 1 \end{array},$$

the upper line is a non-trivial extension of $\pi_1(V)$ by \mathbb{Z}_2 , and the vertical maps are the abelianizations of the groups, and the lower line is equivalent to the sequence $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightarrow \mathbb{Z}_2$. Since S^1 is $K(\mathbb{Z}, 1)$, there is the following pull back diagram of the covering spaces corresponding to the above diagram

$$\begin{array}{ccc} V & \xrightarrow{p} & V/T \\ \downarrow & & \downarrow f \\ S^1 & \xrightarrow{2} & S^1 \end{array},$$

where $S^1 \xrightarrow{2} S^1$ is the usual 2-fold covering and f is a map representing a generator of $H^1(V/T)$. We may assume that f is t -regular at $1 \in S^1$. $f^{-1}(1)$ is a surface in V/T such that $f^{-1}(1) \cap \partial(V/T) = \partial(f^{-1}(1))$ is a simple closed curve in $\partial(V/T)$. Since both $p(D)$ and $\partial(f^{-1}(1))$ are null-homologous in V/T and $\partial(V/T) = S^1 \times S^1$, $p(D)$ and $f^{-1}(1)$ are homo-

topic, and hence isotopic in $\partial(V/T)$. Therefore we may assume that $\partial f^{-1}(1) = p(D)$. Let B be the surface which is the connected component of $p^{-1}(f^{-1}(1))$ containing D as its boundary. Then B satisfies the conditions requested.

Let B be such a surface as in Lemma 2.4. Then $B \cup TB$ decomposes V as $V = A \cup TA$, $A \cap TA = B \cup TB$, where A is a 3-dimensional submanifold of V . By Mayer-Vietoris theorem, we see that $H_1(B \cup TB) = H_1(A) \oplus H_1(TA)$. Let K be the kernel of the homomorphism $H_1(B \cup TB) \rightarrow H_1(A)$ induced by the inclusion. Then $H_1(B \cup TB) = K \div T_*K$, where $T_*: H_1(B \cup TB) \rightarrow H_1(B \cup TB)$ is induced by T . Let $\langle \ , \ \rangle$ be the bilinear form on K defined by $\langle x, y \rangle = x \cdot T_*y$ for $x, y \in K$. Then $\langle \ , \ \rangle$ is a symmetric bilinear form of even type on K . Let $\sigma_{(C, D, B)}(V, T)$ be the signature of this symmetric bilinear form.

Now, let T_0 be the involution on $S^1 \times D^2$ defined by $T_0(x, y) = (-x, y)$ for $(x, y) \in S^1 \times D^2$. By Lemma 2.2, there is an orientation reversing equivariant diffeomorphism $h: \partial V \rightarrow \partial(S^1 \times D^2)$ such that h maps D onto the curve $E = \{(x, x) \mid x \in S^1\} \subset \partial(S^1 \times D^2)$. Then $M_h = V \cup_h S^1 \times D^2$ is a homology sphere with free involution T .

Lemma 2.5. *Let M_h be as above. Then $\sigma_{(C, D, B)}(V, T) = \alpha(M_h, T)$, where α is the Browder Livesay invariant.*

Proof. There is an invariant annulus $S^1 \times I$ in $S^1 \times D^2$ such that $S^1 \times 0 = E$ and $S^1 \times 1 = T_0 E$. Put $\bar{B} = B \cup_E S^1 \times I \cup_{T_0 E} TB$. Then \bar{B} is a characteristic surface of M_h , and M_h decomposes as $M_h = \bar{A} \cup T\bar{A}$, $\bar{A} \cap T\bar{A} = \bar{B}$, where \bar{A} is a submanifold of M_h such that $\bar{A} \cap V = A$ (A is as above). Clearly $H_1(\bar{B}) = H_1(B \cup TB)$ and $\text{Ker}(H_1(\bar{B}) \rightarrow H_1(\bar{A})) = \text{Ker}(H_1(B \cup TB) \rightarrow H_1(A)) = K$. By the definition of $\alpha(M_h, T)$ and $\sigma_{(C, D, B)}(V, T)$, Lemma 2.5 follows.

By the above lemma, $\sigma_{(C, D, B)}(V, T)$ does not depend on the choice of B . Now let (C', D') be another pair of characteristic curves in ∂V of type $(1, 0)$. Let $\bar{h}: \partial(V/T) \rightarrow \partial(S^1 \times D^2/T_0)$ be the map induced by the above equivariant map h . Since $p(D)$ and $p(D')$ are homotopic, and hence isotopic in $\partial V/T$, there is a diffeomorphism $\bar{h}': \partial V/T \rightarrow \partial(S^1 \times D^2/T_0)$ such that \bar{h}' is isotopic to \bar{h} and \bar{h}' maps $p(D')$ onto $p_0(E)$, where p and p_0 are the covering projections of V and $S^1 \times D^2$ respectively. Let h' be the map which covers \bar{h}' . Then h' is equivariantly isotopic to h and $h'(D) = E$. Now form the manifold $M_{h'}$ as above. Then $\alpha(M_h, T) = \alpha(M_{h'}, T)$. This shows that $\sigma_{(C, D, B)}(V, T)$ does not depend on the

particular choice of (C, D) of type $(1, 0)$. Hence the following definition is possible.

Definition 2.3. Let V be a homology circle with orientation preserving involution T . Define $\sigma(V, T) = \sigma_{(C, D, B)}(V, T)$, where (C, D, B) is as above.

Next we consider a pair of characteristic curves in ∂V of type (m, n) , (C, D) , where m is odd > 0 and $n \neq 0$. Since $D \cup TD$ is a characteristic submanifold of ∂V , the relative transversality theorem shows that there is an invariant surface B in V such that $B = TB$ and $\partial B = D \cup TD$, and B decomposes V as $V = A \cup TA$, $A \cap TA = B$, where A is a 3-dimensional submanifold in V . Let K be the kernel of the homomorphism $H_1(B) \rightarrow H_1(A)$ induced by the inclusion. Let $\langle \cdot, \cdot \rangle$ be the bilinear form on K defined by $\langle x, y \rangle = x \cdot T_* y$ for $x, y \in K$. Let $\sigma_{(C, D, B)}(V)$ be the signature of this symmetric bilinear form. First we consider the case with $(V, T) = (S^1 \times D^2, T_0)$, where T_0 is as before.

Lemma 2.6. Let (C, D) be a pair of characteristic curves in $\partial(S^1 \times D^2)$ of type (m, n) , where $n \neq 0$. Let B be a characteristic surface of $S^1 \times D^2$ such that $\partial B = D \cup TD$. Then $\sigma_{(C, D, B)}(S^1 \times D^2, T_0) = \alpha(L(n, m), T)$, where α is the Browder-Livesay invariant, $L(n, m)$ is the standard Lens space and T is the involution on $L(n, m)$ with orbit space $L(2n, m)$.

Proof. Let $f: \partial(S^1 \times D^2) \rightarrow \partial(S^1 \times D^2)$ be an equivariant diffeomorphism such that f maps $S^1 \times 1$ onto C orientation preservingly and $-(1 \times S)$ onto D orientation reversingly. Since (C, D) is of type (m, n) , the manifold $S^1 \times D^2 \cup_f S^1 \times D^2$ is equivariantly diffeomorphic to $L(n, m)$ with the above involution T . Now $\bar{B} = (1 \times D^2) \cup_{(1 \times S^1)} B \cup_{(-1 \times S^1)} (-1 \times D^2)$ is a characteristic surface of $L(n, m)$, and there is a 3-dimensional submanifold \bar{A} of $L(n, m)$ such that $L(n, m) = \bar{A} \cup T\bar{A}$, $\bar{A} \cap T\bar{A} = \bar{B}$ and $\bar{A} = (\bar{A} \cap (S^1 \times D^2)) \cup_f (\bar{A} \cap (S^1 \times D^2)) = I \times D^2 \cup_f A$, where A is as above. Clearly $\text{Ker}(H_1(\bar{B}) \rightarrow H_1(\bar{A})) = \text{Ker}(H_1(B) \rightarrow H_1(A))$. By the definition of $\sigma_{(C, D, B)}(S^1 \times D^2)$, we obtain the lemma.

Lemma 2.7. Let V be a homology circle with orientation preserving free involution T . Let (C, D) be a pair of characteristic curves in ∂V of type (m, n) , where m is odd > 0 and $n \neq 0$. Let B be a characteristic surface in V such that $\partial B = D \cup TD$. Then

$$\sigma_{(C, D, B)}(V, T) = \sigma(V, T) + \alpha(L(n, m), T),$$

where $L(n, m)$ is the standard Lens space and T is the free involution on $L(n, m)$ with orbit space $L(2n, m)$.

Proof. Let $(S^1 \times D^2, T_0)$ be as before. Let $h: \partial(S^1 \times D^2) \rightarrow \partial V$ be an orientation reversing equivariant diffeomorphism such that h maps $S^1 \times 1$ onto C orientation preservingly and $-(1 \times S^1)$ onto D orientation reversingly. Let M be the manifold $(S^1 \times D^2) \cup_r V$. M has a free involution T , and by the same argument in the proof of Lemma 2.6 we obtain $\alpha(M, T) = \sigma_{(C, D, B)}(V, T)$. Now let (C', D') be a pair of characteristic curves in ∂V of type $(1, 0)$. Let B' be a surface in V such that $B' \cap TB' = \emptyset$, $\partial B' = B' \cap \partial V = D'$ and $H_1(B') \rightarrow H_1(V)$ is a zero-homomorphism, where the last map is induced by inclusion (see Lemma 2.4). Put $C_0 = h^{-1}(C')$ and $D_0 = h^{-1}(D')$. Then it can be seen that (C_0, D_0) is a pair of characteristic curves in $\partial(S^1 \times D^2)$ of type $(-r, n)$, where r is an odd integer such that $mr \equiv -1 \pmod{n}$. Let B_0 be a characteristic surface in $S^1 \times D^2$ such that $\partial B_0 = D_0 \cup T_0 D_0$. Then $\bar{B} = B_0 \cup B' \cup TB'$ is a characteristic surface in M . There is a 3-dimensional submanifold \bar{A} with boundary $\partial \bar{A} = \bar{B}$ in M such that $M = \bar{A} \cup T\bar{A}$ and $\bar{A} \cap T\bar{A} = \bar{B}$. Put $A_0 = \bar{A} \cap (S^1 \times D^2)$ and $A' = \bar{A} \cap V$. Then $A_0 \cup TA_0 = S^1 \times D^2$, $A_0 \cap TA_0 = B_0$, $A' \cup TA' = V$ and $A' \cap TA' = B' \cup TB'$. Now it follows that $\text{Ker}(H_1(\bar{B}) \rightarrow H_1(\bar{A})) = \text{Ker}(H_1(B_0) \rightarrow H_1(A_0)) + \text{Ker}(H_1(B' \cup TB') \rightarrow H_1(A'))$. Hence we see that $\alpha(M, T) = \sigma_{(C_0, D_0, B_0)}(S^1 \times D^2, T_0) + \sigma_{(C', D', B')}(V, T)$. By definition, $\sigma_{(C', D', B')}(V, T) = \sigma(V, T)$. By Lemma 2.6, $\sigma_{(C_0, D_0, B_0)}(S^1 \times D^2, T_0) = \alpha(L(n, -r), T)$, where T is the free involution on $L(n, -r)$ with orbit space $L(2n, -r)$. Since $mr \equiv -1 \pmod{n}$, $L(n, -r)$ is equivariantly and orientation preservingly diffeomorphic to $L(n, m)$. Hence $\alpha(L(n, -r), T) = \alpha(L(n, m), T)$. Therefore we obtain $\alpha(M, T) = \sigma(V, T) + \alpha(L(n, m), T)$. This implies that $\sigma_{(C, D, B)}(V, T) = \sigma(V, T) + \alpha(L(n, m), T)$.

Definition 2.3. For a homology circle V with orientation preserving free involution T , we define $\sigma_{(m, n)}(V, T) = \sigma_{(C, D, B)}(V, T)$, where (C, D) is a pair of characteristic curves in ∂V of type (m, n) and B is a characteristic surface in V such that $\partial B = D \cup TD$.

Following [11], we use the following notation: $c(p, q)$ is defined for coprime integer pairs (p, q) with p odd by the recursions

$$\begin{aligned} c(p, \pm 1) &= 0 \\ c(p, -q) &= c(-p, q) = -c(p, q) \\ c(p, p+q) &= c(p, q) + \text{sign}(q(p+q)) \\ c(p+2q, q) &= c(p, q). \end{aligned}$$

In [11], it is claimed that $c(p, q) = \alpha(L(q, p), T)$, where T is the free involution on $L(q, p)$ with orbit space $L(2q, p)$.

Now, by Lemmas 2.7 and 2.5, we readily obtain the following

Theorem 2. *For a homology circle V with orientation preserving free involution T and for a coprime integer pair (m, n) with m odd,*

$$\sigma_{(m, n)}(V, T) = \sigma(V, T) + c(m, n).$$

Moreover, $\sigma(V, T) \equiv 0 \pmod{8}$.

Proposition 1. *Let V_1 and V_2 be homology circles with free orientation preserving involutions T_1 and T_2 , respectively. Let $h: \partial V_1 \rightarrow \partial V_2$ be an orientation reversing equivariant diffeomorphism such that $M = V_1 \cup_h V_2$ is a homology sphere. Then*

$$\alpha(M, T) = \sigma(V_1, T_1) + \sigma(V_2, T_2).$$

Proof. Let (C, D) be a pair of characteristic curves in ∂V_1 of type $(1, 0)$. Then as M is a homology sphere, $(h(C), h(D))$ is a pair of characteristic curves in ∂V_2 of type $(m, \pm 1)$ for some odd integer m . By the same argument in the proof of Lemma 2.7, we see that $\alpha(M, T) = \sigma(V_1, T_1) + \sigma_{(m, \pm 1)}(V_2, T_2)$. Now $\sigma_{(m, \pm 1)}(V_2, T_2) = \sigma(V_2, T_2) + c(m, \pm 1) = \sigma(V_2, T_2)$.

Some examples. (1) Let T be the antipodal involution on S^3 . Let K be a knot in S^3 such that $TK = K$. Let N be an invariant closed tubular neighborhood of K in S^3 . Let $V(K)$ be the closure of the complement of N , $V(K) = \overline{S^3 - N}$. Then $V(K)$ is a homology circle with free involution T .

Proposition 2. $\sigma(V(K), T) = 0$.

Proof. Since $S^3 = V(K) \cup N$, $\alpha(S^3, T) = \sigma(V(K), T) + \sigma(N, T)$ by Proposition 1. Now $\alpha(S^3, T) = \sigma(N, T) = 0$.

(2) Let K be a knot in S^3 such that $\Delta(-1) = \pm 1$, where $\Delta(t)$ is the Alexander polynomial of K . Let M be the 2-fold branched covering space of S^3 branched over K . Then M is a homology sphere with involution. Let N be an invariant closed tubular neighborhood of K in M . Let V_K be the closure of the complement of N , $V_K = \overline{M - N}$. Then V_K is a homology circle with free involution T .

Proposition 3. $\sigma(V_K, T) = -\sigma(K)$, where $\sigma(K)$ is the signature of the knot K .

Proof. Let $(S^1 \times D^2, T_0)$ be as before. Let $h: \partial(S^1 \times D^2) \rightarrow \partial V_K$ be an orientation reversing equivariant diffeomorphism such that $(h(S^1 \times 1), h(1 \times S^1))$ is a pair of characteristic curves in ∂V of type $(m, \pm 1)$ for some odd integer m . Then $H = S^1 \times D^2 \cup_h V$ is a homology sphere with free involution T . By Proposition 1, $\alpha(H, T) = \sigma(V, T)$. Now the process of constructing H from the knot K in S^3 is just the converse of the process of constructing the knot $K_1 = (M_1, C_1)$ in the proof of Theorem 1. By Remark 1, $\alpha(H, T) = -\sigma(K)$. Hence $\sigma(V_K, T) = -\sigma(K)$.

3. Browder-Livesay invariants of some homology spheres. In this section, first we consider the homology circles with free involution constructed as follows:

Let V_i ($i=1, \dots, k$) be a homology circle with orientation preserving free involution T_i . Let (C_i, D_i) be a pair of characteristic curves in ∂V_i of type (m_i, n_i) ($i=1, \dots, k$). Let T_0 be the free involution on $S^1 \times D^2$ defined by $T_0(x, y) = (-x, y)$ for $(x, y) \in S^1 \times D^2$. Let $\{p_i\}_{i=1, \dots, k}$ be k distinct points of ∂D^2 and let E_i be an arc in ∂D^2 containing p_i in its interior such that $E_i \cap E_j = \emptyset$ for each $i \neq j$ ($i, j=1, \dots, k$). Let $V((m_1, n_1), \dots, (m_k, n_k))$ be the 3-manifold obtained from the disjoint union $(S^1 \times D^2) \cup (\bigcup_{i=1}^k V_i)$ by identifying each $S^1 \times E_i$ with an invariant annular neighborhood of C_i in ∂V_i by an equivariant diffeomorphism f_i which maps $S^1 \times p_i$ onto C_i in an orientation preserving manner and reverses orientation on a transverse arc.

Lemma 3.1. If m_i and m_j are coprime for each $i \neq j$ ($i, j=1, \dots, k$), then $V((m_1, n_1), \dots, (m_k, n_k))$ is a homology circle.

Proof. There are integers r_1, \dots, r_k such that $\sum m_1 \cdots m_{i-1} r_i m_{i+1} \cdots m_k = 1$. Let λ_i be a generator of $H_1(V_i)$ such that $[C_i] = m_i \lambda_i$ ($i=1, \dots, k$). By Mayer Vietoris theorem, we see that $r_1 \lambda_1 + \cdots + r_k \lambda_k$ generates $H_1(V((m_1, n_1), \dots, (m_k, n_k))) = \mathbb{Z}$.

Since each f_i is equivariant, $V((m_1, n_1), \dots, (m_k, n_k))$ has a free involution T .

Theorem 3. Let $V = V((m_1, n_1), \dots, (m_k, n_k))$ be the above manifold, where m_i and m_j are coprime for each $i \neq j$ ($i, j=1, \dots, k$). Then

$$\sigma(V, T) = \sum_{i=1}^k (\sigma(V_i, T_i) + c(m_i, n_i) + \text{sign } n_i) - c\left(m_1 \cdots m_k, m_1 \cdots m_k \sum_{i=1}^k \frac{n_i}{m_i}\right) - \text{sign} \left(\sum_{i=1}^k \frac{n_i}{m_i} \right),$$

where σ and c are as in § 2, and $c(m, 0) = \text{sign } 0 = 0$.

Corollary. Let h be an orientation reversing equivariant diffeomorphism from $\partial(S^1 \times D^2)$ to ∂V such that $(S^1 \times D^2) \cup_h V$ is a homology sphere with free involution T , where V is as in Theorem 3. Then $\alpha((S^1 \times D^2) \cup_h V, T)$ is given by the right hand term of the equation in Theorem 3.

Proof. This is straightforward from Proposition 1 in § 2 and Theorem 3.

Remark. When $(V_i, T_i) = (S^1 \times D^2, T_0)$ for $i = 1, \dots, k$ in the above construction, it can be seen that $V = V((m_1, n_1), \dots, (m_k, n_k))$ has an S^1 action which extends the involution. Moreover in the case that m_i and m_j are coprime for each $i \neq j$ ($i, j = 1, \dots, k$) and $m_1 \cdots m_k \left(\sum_{i=1}^k \frac{n_i}{m_i} \right) = \pm 1$, by a suitable choice of h in the above Corollary, the resulting manifold $S^1 \times D^2 \cup_h V = M$ may be endowed with S^1 action which extends the involution, that is, this manifold is a Seifert manifold and $(m_1, n_1), \dots, (m_k, n_k)$ are its Seifert invariants. In this case the above formula of $\alpha(V, T)$ gives the formula of W. D. Neumann [10],

$$\alpha(M, T) = \sum_{i=1}^k (c(m_i, n_i) + \text{sign } n_i) - \text{sign} \left(\sum_{i=1}^k \frac{n_i}{m_i} \right).$$

Hence the above formula is a generalization of Neumann's formula for the Browder-Livesay invariants of Seifert homology spheres.

Proof of Theorem 3. We prove Theorem 3 by the induction on k . When $k = 1$, Theorem 3 is trivial since V is equivariantly diffeomorphic to V_1 and $\text{sign } n_1 = \text{sign } n_1/m_1$ ($m_1 > 0$). Next we consider the case with $k = 2$. The formula which we must prove is

$$\sigma(V, T) = \sigma(V_1, T_1) + \sigma(V_2, T_2) + c(m_1, n_1) + c(m_2, n_2) - c(m_1 m_2, m_1 n_2 + m_2 n_1) + \text{sign } n_1 + \text{sign } n_2 - \text{sign } (n_1/m_1 + n_2/m_2).$$

Let B_i be a characteristic surface of V_i such that $\partial B_i = D_i \cup T D_i$ ($i = 1, 2$). If $n_i = 0$, then we take $B \cup T B$ as B_i where B is a surface in V_i satisfying the conditions of Lemma 2.4. We may assume that $f_i^{-1}(D_i) = 1 \times E_i \subset \partial(S^1 \times D^2)$ and $f_i^{-1}(T D_i) = -1 \times E_i$. Then $\bar{B} = B_1 \cup B_2 \cup 1 \times D^2 \cup -1 \times D^2$

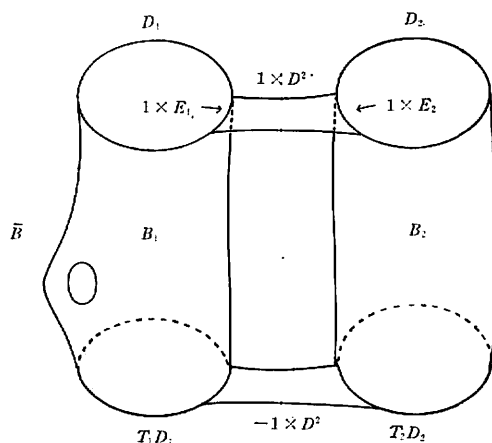


Figure 1

(where $B_i \subset V_i$ ($i=1, 2$) and $\{\pm 1\} \times D^2 \subset S^1 \times D^2$) is a characteristic surface of V (Figure 1).

Put $\bar{D}_1 \cup \bar{D}_2 \cup 1 \times \partial D^2 = 1 \times (E_1 \cup E_2)$

$= D$. Let r_1 and r_2 be integers such that $r_1 m_2 + r_2 m_1 = 1$.

Then $\lambda = r_1 \lambda_1 + r_2 \lambda_2$ generates $H_1(V) = \mathbb{Z}$, and $\lambda_1 = m_2 \lambda$ and $\lambda_2 = m_1 \lambda$. Now $[D_1] = n_1 \lambda_1 = n_1 m_2 \lambda$ and $[D_2] = n_2 \lambda_2 = n_2 m_1 \lambda$ in $H_1(V)$. Therefore $[D] = (n_1 m_2 + n_2 m_1) \lambda$ in $H_1(V)$. Let C be the curve $S^1 \times p \subset \partial(S^1 \times D^2)$,

where $p \in D^2 - \bigcup_{i=1}^k E_k$. Then

$C \subset \partial V$. Clearly $C \cap D$ consists of a single point, and $[C] = m_1 \lambda_1 = m_1 \lambda_2 = m_1 m_2 \lambda$ in $H_1(V)$. Hence (C, D) is a pair of characteristic curves in ∂V of type $(m_1 m_2, m_1 n_2 + m_2 n_1)$. Therefore $\sigma_{(C, D, B)}(V, T) = \sigma_{(C_1, D_1, B_1)}(V_1, T_1) + \sigma_{(C_2, D_2, B_2)}(V_2, T_2) + \varepsilon$,

Assertion (1). $\sigma_{(C, D, B)}(V, T) = \sigma_{(C_1, D_1, B_1)}(V_1, T_1) + \sigma_{(C_2, D_2, B_2)}(V_2, T_2) + \varepsilon$, where $\varepsilon = \pm 1$ if $n_1 n_2 \neq 0$, and $\varepsilon = 0$ if $n_1 n_2 = 0$.

Proof. If $n_i \neq 0$, then the curve D_i is not homologous to zero in V_i and there is a connected component in B_i with boundary $D_i \cup T D_i$, and if $n_i = 0$, then $B_i = B \cup T B$ where B is a surface such that $\partial B = D_i$ and $\partial T B = T D_i$ ($i=1, 2$). Hence by Mayer-Vietoris theorem, it follows that $\text{rank } H_1(B) = \text{rank } H_1(B_1) + \text{rank } H_1(B_2) + 2$ if $n_1 n_2 \neq 0$, and $H_1(B) \cong H_1(B_1) + H_1(B_2)$ if $n_1 n_2 = 0$. Let \bar{A} be a 3-dimensional submanifold of V such that $V = \bar{A} \cup T \bar{A}$, $\bar{A} \cap T \bar{A} = \bar{B}$. Put $A_i = \bar{A} \cap V_i$ and $A_2 = \bar{A} \cap V_2$. Then $A_i \cup T A_i = V_i$ and $A_i \cap T A_i = B_i$ ($i=1, 2$). Put $K = \text{Ker } (H_1(\bar{B}) \rightarrow H_1(\bar{A}))$ and put $K_i = \text{Ker } (H_1(B_i) \rightarrow H_1(A_i))$ ($i=1, 2$), where all the homomorphisms are induced by the inclusions. Then we see that $\text{rank } K = \text{rank } K_1 + \text{rank } K_2 + 1$ if $n_1 n_2 \neq 0$, and $K = K_1 + K_2$ if $n_1 n_2 = 0$. Now $K \otimes \mathbb{Q}$ is an inner-product space by the bilinear form defined by $\langle x, y \rangle = x \cdot T_* y$ for $x, y \in K \otimes \mathbb{Q}$, where \mathbb{Q} is the rational number field, and $K_1 \otimes \mathbb{Q} + K_2 \otimes \mathbb{Q}$ is a sub-inner-product space in $K \otimes \mathbb{Q}$. If $n_1 n_2 \neq 0$, then by the orthogonal decomposition theorem ([9]), $K \otimes \mathbb{Q}$ is isomorphic to an orthogonal sum $K_1 \otimes \mathbb{Q} + K_2 \otimes \mathbb{Q} + I$, where I is a 1-dimensional inner-product space over \mathbb{Q} . This proves Assertion (1).

By Assertion (1) and Theorem 2, we obtain

$$\begin{aligned}\sigma(V, T) = & \sigma(V_1, T_1) + \sigma(V_2, T_2) + c(m_1, n_1) + c(m_2, n_2) \\ & - c(m_1 m_2, m_1 n_2 + m_2 n_1) + \varepsilon,\end{aligned}$$

where $\varepsilon = \pm 1$ if $n_1 n_2 \neq 0$, and $\varepsilon = 0$ if $n_1 n_2 = 0$. Now we assume that $n_1 n_2 \neq 0$. We must determine ε . Unfortunately we cannot determine ε geometrically. We use an algebraic condition as follows: By Theorem 2, $\sigma(V, T) \equiv \sigma(V_1, T_1) \equiv \sigma(V_2, T_2) \equiv 0 \pmod{8}$. Hence $c(m_1, n_1) + c(m_2, n_2) - c(m_1 m_2, m_1 n_2 + m_2 n_1) + \varepsilon \equiv 0 \pmod{8}$. Now Hirzebruch ([6]) has shown the following formula

$$c(p, q) \equiv 1 - 2\left(\frac{p}{q}\right) + q \pmod{8} \text{ for } q \text{ odd } < 0,$$

where $\left(\frac{p}{q}\right)$ is the quadratic residue symbol. This formula implies the following

Assertion (2). *If n is odd, $c(m, n) \equiv n + \text{sign } n \pmod{4}$, and if n is even, $c(m, n) \equiv m + n + (1 + \text{sign } n) \pmod{4}$.*

Proof. Since $-2\left(\frac{p}{q}\right) \equiv 2 \pmod{4}$ and $c(p, -q) = -c(p, q)$, Hirzebruch's formula gives the first equation for n odd. For n even, use the relation $c(m, n) = c(m, m+n) - \text{sign } (n(m+n))$ (note that m is odd).

By Assertion (2), a brief calculation shows that

$$\begin{aligned}& -c(m_1 m_2, m_1 n_2 + m_2 n_1) + c(m_1, n_1) + c(m_2, n_2) + \varepsilon \\ & \equiv (\text{sign } n_1 + \text{sign } n_2 - \text{sign } (m_1 n_2 + m_2 n_1)) + 2 + \varepsilon \pmod{4}.\end{aligned}$$

Since this must be $0 \pmod{4}$ and both ε and $\text{sign } n_1 + \text{sign } n_2 - \text{sign } (m_1 n_2 + m_2 n_1)$ are ± 1 ($m_1 m_2 > 0$), we obtain $\varepsilon = \text{sign } n_1 + \text{sign } n_2 - \text{sign } (m_1 n_2 + m_2 n_1)$. This completes the proof of Theorem 3 in the case with $k=2$. Now by the inductive calculation, we obtain the theorem.

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