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INVARIANT MEASURES AND ERGODIC THEOREMS FOR POSITIVE OPERATORS ON $C(X)$ WITH X QUASI-STONIAN

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1. Introduction. Let X be a quasi-Stonian compact Hausdorff space. Relations between invariant measures and ergodic theorems are investigated for a positive σ-additive linear operator T on $C(X)$ satisfying $\sup_n \|(1/n)\sum_{i=0}^{n-1} T^i\| < \infty$. Such relations were investigated by Ando [2] for a σ-additive Markov operator T on $C(X)$. In this paper Ando's results will be generalized.

2. Definitions and theorems. For a topological space X, let $C(X)$ denote the Banach space of real-valued bounded continuous functions on X with the supremum norm. In this paper, unless the contrary is explicitly explained, X is assumed to be a compact Hausdorff space which is quasi-Stonian, i. e. to each bounded sequence (f_n) in $C(X)$ there corresponds an $f \in C(X)$, with $f \leq f_n$ for all $n \geq 1$, such that if $g \in C(X)$ and $g \leq f_n$ for all $n \geq 1$ then $g \leq f$. This function f is called the *infimum* of (f_n) and denoted by

$$f = \bigwedge_{n=1}^{\infty} f_n.$$

The function $g \in C(X)$ defined by $g = -\bigwedge_{n=1}^{\infty}(-f_n)$ is called the *supremum* of (f_n) and denoted by

$$g = \bigvee_{n=1}^{\infty} f_n.$$

Clearly, g is minimal with respect to the conditions $g \in C(X)$ and $f_n \leq g$ for all $n \geq 1$. As in Ando [2], we let

$$\text{(O)-}\lim_n \sup f_n = \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} f_k$$

and

$$\text{(O)-}\lim_n \inf f_n = \bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} f_k.$$

If (O)-$\lim_n \sup f_n =$ (O)-$\lim_n \inf f_n$ holds, then we denote by (O)-$\lim_n f_n$ these equal functions.

As usual, the dual space $C^*(X)$ is identified with the space $M(X)$ of all bounded regular Borel measures on X by the relation

$$\langle f, \mu\rangle=\int f\,d\mu \qquad (f\in C(X),\ \mu\in M(X)).$$

$\mu\in M(X)$ is called *σ-additive* if $f_n\in C(X)$, $f_n\geq f_{n+1}\geq 0$ for each $n\geq 1$ and $\bigwedge_{n=1}^{\infty} f_n=0$ imply $\lim_n \langle f_n, |\mu|\rangle=0$, where $|\mu|$ denotes the total variation of μ.

Let T be a positive linear operator on $C(X)$. T is called a *contraction operator* if $\|T\|\leq 1$, a *Markov operator* if $\|T\|=1=T1$, and *σ-additive* if $\bigwedge_{n=1}^{\infty} Tf_n=0$ whenever $f_n\in C(X)$, $f_n\geq f_{n+1}\geq 0$ for each $n\geq 1$, and $\bigwedge_{n=1}^{\infty} f_n=0$.

In this paper the following theorems are proved. In case T is a σ-additive Markov operator on $C(X)$, these theorems are due to Ando [2].

Theorem 1. *Let X be a quasi-Stonian compact Hausdorff space and T a positive σ-additive linear operator on $C(X)$ satisfying*

$$\sup_n \left\|\frac{1}{n}\sum_{i=0}^{n-1} T^i\right\|<\infty.$$

Then the following statements are equivalent:

(a) *All invariant measures (with respect to T^*) are σ-additive.*

(b) *For each $f\in C(X)$ $\frac{1}{n}\sum_{i=0}^{n-1} T^i f$ converges in norm, and further* $\dim\,(N_T)<\infty$ *where* $N_T=\{f\in C(X):\ Tf=f\}$.

Theorem 2. *Let X and T be as in Theorem 1. Suppose the space $M_\sigma(X)$ of all σ-additive measures is weak*-dense in $M(X)$. Then the following statements are equivalent:*

(a) *For each $\mu\in M_\sigma(X)$ $\frac{1}{n}\sum_{i=0}^{n-1} T^{*i}\mu$ converges in norm.*

(b) *For each $f\in C(X)$* (O)-$\lim_n \frac{1}{n}\sum_{i=0}^{n-1} T^i f$ *exists.*

Theorem 3. *Let X and T be as in Theorem 1. Suppose there exists a strictly positive σ-additive measure. Then the following statements are equivalent:*

(a) *There exists a strictly positive σ-additive invariant measure (with repect to T^*).*

(b) *If $0\leq f\in C(X)$ and $f\neq 0$, then* (O)-$\lim\sup_n \frac{1}{n}\sum_{i=0}^{n-1} T^i f\neq 0$.

Remark 1. Let X be any compact Hausdorff space and T any positive linear operator on $C(X)$ satisfying $\sup_n \|\frac{1}{n}\sum_{i=0}^{n-1} T^i\| < \infty$. Then, for each $f \in C(X)$, $\frac{1}{n}\sum_{i=0}^{n-1} T^i f$ converges in norm if and only if N_T separates N_T^*, where $N_T^* = \{\mu \in M(X): T^*\mu = \mu\}$. This follows from [9], as $\lim_n \|T^n\|/n = 0$, which is due essentially to Derriennic and Lin [3] and will be shown in the next section.

Remark 2. Let X and T be as in Theorem 1. If, for each $f \in C(X)$, $\frac{1}{n}\sum_{i=0}^{n-1} T^i f$ converges in norm, then, for each $\mu \in M(X)$, $\frac{1}{n}\sum_{i=0}^{n-1} T^{*i}\mu$ converges in norm, too. This follows from [1] and a mean ergodic theorem (see, for example, Theorem VIII. 5. 1 in [4]).

3. A lemma. The purpose of this section is to prove the following lemma, which is fundamental throughout the remainder.

Lemma. *Let X be a compact Hausdorff space and T a positive linear operator on $C(X)$ satisfying* $\sup_n \|\frac{1}{n}\sum_{i=0}^{n-1} T^i\| = M < \infty$.

Then

$$\lim_n \|T^n\|/n = 0.$$

If, in addition, X is quasi-Stonian and T is σ-additive, then there exists a function $s \in C(X)$ such that

(i) $Ts = s \geq 0$.

(ii) *if $f \in C(X)$ satisfies $\|f\| \leq 1$ and $Tf = f$ then $|f| \leq s$,*

(iii) *if $\mu \in M_\sigma(X)$ satisfies* $\operatorname{supp} \mu \subset \{x : s(x) = 0\}$ *then*

$$\lim_n \frac{1}{n}\sum_{i=0}^{n-1} \|T^{*i}\mu\| = 0.$$

Proof. Given an $n \geq 1$ and an $\varepsilon > 0$, choose $0 \leq \mu \in M(X)$ so that $\|\mu\| = 1$ and $\|T^{*n}\mu\| > \|T^{*n}\| - \varepsilon$ $(= \|T^n\| - \varepsilon)$. Then, for any $0 \leq k \leq n$,

$$(k+1)^{-1}\|T^n\| < \varepsilon + (k+1)^{-1} \sum_{i=n-k}^{n} \|T^{*i}\mu\| \leq \varepsilon + M\|T^{*n-k}\mu\|.$$

Therefore, as in Derriennic and Lin [3], we get

$$(\|T^n\|/n) \sum_{k=0}^{n-1} (k+1)^{-1} \leq \varepsilon + \frac{M}{n}\sum_{k=0}^{n-1} \|T^{*n-k}\mu\|$$
$$\leq \varepsilon + M^2 \|T^*\mu\| \leq \varepsilon + M^2 \|T\|.$$

Hence $\lim_n \|T^n\|/n=0$, as $\lim_n \sum_{k=0}^{n-1}(k+1)^{-1}=\infty$.

To prove the second part of the lemma, let, as in [3] and [7],

$$t = \text{(O)-}\limsup_n \frac{1}{n}\sum_{i=0}^{n-1} T^i 1.$$

Since T is σ-additive,

$$Tt \geq \text{(O)-}\limsup_n \frac{1}{n}\sum_{i=1}^{n} T^i 1 \geq t.$$

Thus if we set

$$s=\text{(O)-}\lim_n T^n t \ (= \text{(O)-}\lim_n \frac{1}{n}\sum_{i=0}^{n-1} T^i t),$$

then

$$Ts=s\geq t\geq 0.$$

(ii) is immediate from the definition of s.

For the proof of (iii), it is enough to prove it for $0\leq\mu\in M_\sigma(X)$ with supp $\mu\subset\{x:s(x)=0\}$. Then, since μ is σ-additive, we get

$$\begin{aligned}\limsup \frac{1}{n}\sum_{i=0}^{n-1}\|T^{*i}\mu\| &= \limsup_n \int \frac{1}{n}\sum_{i=0}^{n-1} T^i 1\, d\mu \\ &\leq \lim_n \int \Big(\bigvee_{k=n}^{\infty}\frac{1}{k}\sum_{i=0}^{k-1} T^i 1\Big)\, d\mu \\ &= \int \text{(O)-}\limsup_n \frac{1}{n}\sum_{i=0}^{n-1} T^i 1\, d\mu \\ &= \int t\, d\mu \leq \int s\, d\mu = 0,\end{aligned}$$

and hence (iii) is proved. The proof is complete.

4. Proof of Theorem 1. The following argument is a modification of the proof of Theorem 2 in Ando [2].

Let $s\in C(X)$ be as in the above lemma, and put

$$Y=\{x:s(x)>0\} \text{ and } Z=\{x:s(x)=0\}.$$

Further, write

$$T_n=\frac{1}{n}\sum_{i=0}^{n-1} T^i \qquad (n\geq 1).$$

(a) $\Longrightarrow$ (b) : Suppose (a) holds. If $T_n f$ does not converge in norm for some $f\in C(X)$, then by a mean ergodic theorem due to Sine [11] (see also Lloyd [6] and the author [9], [10]) there exists a nonzero invariant measure μ (with respect to T^*) such that

$$\text{if } f \in C(X) \text{ and } Tf = f, \text{ then } \langle f, \mu \rangle = 0.$$

Let μ^+ and μ^- denote the positive and negative parts of μ, respectively. Then $\mu = T^*\mu = T^*\mu^+ - T^*\mu^-$ implies that

$$T^*\mu^+ \geq \mu^+ \text{ and } T^*\mu^- \geq \mu^-.$$

Since μ is σ-additive, the above lemma shows that either $Y \cap \operatorname{supp} \mu^+ \neq \emptyset$ or $Y \cap \operatorname{supp} \mu^- \neq \emptyset$ holds. Without loss of generality we may assume that

$$Y \cap \operatorname{supp} \mu^+ \neq \emptyset.$$

Define

$$\varphi^+ = \lim_n T_n^* \mu^- \text{ and } \varphi^- = \lim_n T_n^* \mu^-.$$

Then φ^+ and φ^- are invariant measures (with respect to T^*) satisfying

$$\langle s, \varphi^+ \rangle = \lim_n \langle s, T_n^* \mu^+ \rangle = \langle s, \mu^+ \rangle > 0$$

and

$$\langle s, \varphi^- \rangle = \langle s, \mu^- \rangle.$$

Since $\varphi^+ \geq \mu^+$, it follows that $\varphi^+ = \mu^+$ on Y. Similarly, $\varphi^- = \mu^-$ on Y. Since μ^+ and μ^- are disjoint in the sense of Ando [2], by Lemma 1 in [2] there exists a function $0 \leq f \in C(X)$, with $f \leq s$, such that

$$\langle f, \varphi^+ \rangle = \langle f, \mu^+ \rangle > 0 \text{ and } \langle f, \varphi^- \rangle = \langle f, \mu^- \rangle = 0.$$

Then the function $g = (\mathrm{O})\text{-}\limsup_n T_n f$ satisfies:

$$Tg \geq g \geq 0,$$

$$\begin{aligned} \langle g, \mu^+ \rangle &\geq \limsup_n \langle T_n f, \mu^+ \rangle \\ &= \limsup_n \langle f, T_n^* \mu^+ \rangle \geq \langle f, \mu^+ \rangle > 0, \end{aligned}$$

and

$$\langle g, \varphi^- \rangle \leq \sum_{n=1}^{\infty} \langle T_n f, \varphi^- \rangle = 0.$$

Therefore if we set

$$h = (\mathrm{O})\text{-}\lim_n T_n g,$$

then $Th = h$, $\langle h, \mu^+ \rangle \geq \langle g, \mu^+ \rangle > 0$, and

$$\langle h, \mu^- \rangle \leq \langle h, \varphi^- \rangle = \lim_n \langle T_n g, \varphi^- \rangle = 0.$$

Consequently, $\langle h, \mu \rangle = \langle h, \mu^+ \rangle - \langle h, \mu^- \rangle \neq 0$. But this is a contradiction, since $Th = h$ implies $\langle h, \mu \rangle = 0$. This contradiction shows that for each $f \in C(X)$ $T_n f$ converges in norm.

Define a positive linear operator P on $C(X)$ as

$$Pf = \lim_n T_n f \qquad (f \in C(X)).$$

Since $\dim (N_T) = 0$ whenever $s = 0$, we shall consider only the case $s \neq 0$, below. Then $P \neq 0$, and clearly

$$P^2 = P = PT = TP.$$

Put $\mathscr{J} = \{f \in C(X) : P|f| = 0\}$ and

$$K = \bigcap_f \{x : f(x) = 0\}, \quad \text{where } f \in \mathscr{J}.$$

Then $K \neq \emptyset$, and furthermore

$$\mathscr{J} = \{f \in C(X) : f = 0 \text{ on } K\}.$$

(In fact, if $f = 0$ on K, then by the compactness of X for any $\varepsilon > 0$ there exist finitely many functions $f_1, \cdots, f_n$ in $\mathscr{J}$ and a constant $\delta > 0$ such that

$$|f(x)| \geq \varepsilon \text{ implies } \sum_{i=1}^{n} |f_i(x)| \geq \delta.$$

Then the function $g = \sum_{i=1}^{n} |f_i|$ is in $\mathscr{J}$, and it is easily seen that $P|f| \leq \varepsilon \|P\|$, which proves that $f \in \mathscr{J}$.) Since supp $P^*\delta_x \subset K$ and supp $T^*\delta_x \subset K$ for every $x \in K$, where δ_x denotes the unit point mass concentrated at x, P and T induce, respectively, positive linear operators $P^\sim$ and $T^\sim$ on $C(K)$ by the relations

$$P^\sim f^\sim(x) = \int_K f^\sim(y)\, dP^*\delta_x(y) \qquad (f^\sim \in C(K), x \in K)$$

and

$$T^\sim f^\sim(x) = \int_K f^\sim(y)\, dT^*\delta_x(y) \qquad (f^\sim \in C(K), x \in K).$$

It is easily seen that

$$\dim (N_T) = \dim (N_{T^\sim}) = \dim (N_{P^\sim}).$$

Since $(T^\sim)_n$ converges strongly to $P^\sim$ and since $P^\sim$ is stricly positive on $C(K)$, it follows from a standard argument that if $f^\sim$ and $g^\sim$ are in $N_{T^\sim} (= N_{P^\sim})$ then $\max (f^\sim, g^\sim)$ is also in $N_{T^\sim}$.

For any $x \in X$, $P^*\delta_x$ is an invariant measure (with respect to T^*) and hence σ-additive. It follows that if $f_n \in N_T$, $f_n \geq f_{n+1} \geq 0$ for each $n \geq 1$ and $f = \bigwedge_{n=1}^{\infty} f_n$ then $\lim_n f_n(x) = \lim_n Pf_n(x) = Pf(x)$ for each $x \in X$. Hence, by Dini's theorem, f_n converges in norm to Pf, and therefore $Pf = f$. (This argument is due to Ando.)

Now let us fix an ε, with $0 < \varepsilon < 1$, so that the set

$$F = F(\varepsilon) = K \cap \{x : s(x) \geq \varepsilon\}$$

is not empty, and put

$$N_T(F) = \{f \in C(F) : f = g \text{ on } F \text{ for some } g \in N_T\}$$

and

$$M_T(F) = \{f \in C(F) : f = g/s \text{ on } F \text{ for some } g \in N_T\}.$$

Since $M_T(F)$ is a Banach lattice containing the constant functions, we see without difficulty that if $f, g \in M_T(F)$ then $fg \in M_T(F)$. Moreover, if $f_n \in M_T(F)$ and $f_n \geq f_{n+1} \geq 0$ for each $n \geq 1$, then f_n converges in norm to a function in $M_T(F)$. (To see this, since $s \geq \varepsilon$ on F, it is enough to note that there exists a sequence (g_n) in N_T such that $g_n \geq g_{n+1} \geq 0$ on X and $f_n = g_n/s$ on F for each $n \geq 1$.) We now apply the Gelfand-Naimark theorem to infer that there exists a positive multiplicative linear isometry from $M_T(F)$ onto some $C(Z)$, where Z is a compact Hausdorff space. It follows that if $f_n \in C(Z)$ and $f_n \geq f_{n+1} \geq 0$ for each $n \geq 1$, then f_n converges in norm. But this is possible only if Z is a finite set, and hence we conclude that

$$\dim N_T(F) = \dim M_T(F) = \dim C(Z) = d < \infty.$$

To prove that $\dim (N_{T^\sim}) = d$, fix any basis $\{f_1, \cdots, f_d\}$ of $N_T(F)$. Then any $f^\sim \in N_{T^\sim}$ has the form

$$f^\sim = \sum_{i=1}^{d} a_i f_i \qquad \text{on } F.$$

Thus there exists a function $g^\sim \in N_{T^\sim}$ satifying $g^\sim = 0$ on F. It is then enough to show that such a $g^\sim$ must satify $g^\sim = 0$ on K. Assume the contrary: $g^\sim \neq 0$. Here, without loss of generality, we may assume that

$$\|g^\sim\| = 1 \text{ and } g^\sim = 0 \text{ on } F = F(\varepsilon).$$

Therefore for some $x \in K - F(\varepsilon)$ we get

$$1 = | g^\sim(x) | > s(x).$$

But this is impossible, because

$$1 = |g^\sim(x)| \leq P^\sim |g^\sim|(x) \leq P^\sim 1(x) = s(x) < \varepsilon < 1.$$

(b) $\Longrightarrow$ (a): Suppose (b) holds. We again let $Pf = \lim_n T_n f$ for all

$f \in C(X)$. Suppose μ is an invariant measure with respect to T^*. To prove that μ is σ-additive, it may be assumed without loss of generality that $\mu \geq 0$. Let $f_n \in C(X)$, $f_n \geq f_{n+1} \geq 0$ for each $n \geq 1$, and $\bigwedge_{n=1}^{\infty} f_n = 0$. Then, since

$$Pf_n \geq Pf_{n+1} \geq 0 \text{ and } \langle f_n, \mu \rangle = \langle Pf_n, \mu \rangle$$

for each $n \geq 1$ and since dim $(N_T) < \infty$, Pf_n converges in norm to a function g in N_T and

$$\lim_n \langle f_n, \mu \rangle = \langle g, \mu \rangle.$$

On the other hand, since $T_n^* \delta_x$ converges in norm for each $x \in X$ and since all the T_n are σ-additive, we see, as in [2], pp. 182 — 183, that

$$\lim_n Pf_n(x) = 0$$

on a dense subset of X. Therefore $g = 0$ on X, and $\langle g, \mu \rangle = 0$. This shows that μ is σ-additive, and the proof is completed.

5. **Proof of Theorem 2.** (a) $\Longrightarrow$ (b) : Suppose (a) holds. Let $f \in C(X)$ and $0 \leq \mu \in M_\sigma(X)$. We first show that $\lim_n T_n f(x)$ exists for almost all $x \in X$ with respect to μ. To do this, it may be assumed without loss of generality that $T^* \mu$ is absolutely continuous with respect to μ. (In fact, if necessary, consider the measure $\lambda = \sum_{n=0}^{\infty} (2^n \| T^{*n} \mu \|)^{-1} T^{*n} \mu$ ($\in M_\sigma(X)$) instead of μ. Then, clearly, μ and $T^* \lambda$ are absolutely continuous with respect to λ.) Then T^* can be regarded as a positive linear operator on $L_1(\mu)$, and for each $\xi \in L_1(\mu)$, $T_n^* \xi$ converges in norm. Hence, by Theorem 1 in the author [8], $\lim_n T_n f(x)$ exists almost everywhere with respect to μ. Since μ is σ-additive, it then follows that

$$[(\mathrm{O})\text{-}\limsup_n T_n f](x) = [(\mathrm{O})\text{-}\liminf_n T_n f](x)$$

almost everywhere with respect to μ. Therefore we get

$$(\mathrm{O})\text{-}\limsup_n T_n f = (\mathrm{O})\text{-}\liminf_n T_n f \text{ on supp } \mu,$$

and hence on X, because the set $E = \cup \{\text{supp } \mu : 0 \leq \mu \in M_\sigma(X)\}$ is a dense subset of X, which is an easy consequence of the hypothesis that $M_\sigma(X)$ is weak*-dense in $M(X)$.

(b) $\Longrightarrow$ (a) : Suppose (b) holds. For $\mu \in M_\sigma(X)$ and $f \in C(X)$ we have by the σ-additivity of μ that, for almost all $x \in X$ with respect to μ,

$$\limsup_n T_n f(x) = [(\mathrm{O})\text{-}\limsup_n T_n f](x)$$

and

$$\liminf_n T_n f(x) = [(\mathrm{O})\text{-}\liminf_n T_n f](x).$$

Hence by (b), for almost all $x \in X$ with respect to μ,

$$\lim_n T_n f(x) = [(\mathrm{O})\text{-}\lim_n T_n f](x).$$

We now apply Lebesgue's convergence theorem to infer that

$$\lim_n \langle f, T_n^* \mu \rangle = \lim_n \langle T_n f, \mu \rangle = \langle (\mathrm{O}\text{-}\lim_n T_n f, \mu \rangle.$$

It follows that $T_n^* \mu$ converges in the weak*-topology, and thus by [1] $T_n^* \mu$ converges weakly. Hence the norm convergence of $T_n^* \mu$ follows from a mean ergodic theorem (cf. Theorem VIII. 5. 1 in [4]).

6. Proof of Theorem 3. To prove Theorem 3 we need the following proposition, which is a sharpened form of Theorem 3 in Ando [2].

Proposition. *Let X be a quasi-Stonian compact Hausdorff space and T a positive σ-additive linear contraction operator on $C(X)$. Suppose there exists a strictly positive σ-additive measure μ in $M(X)$. Then the space X decomposes into two open sets P and N such that*

(i) *$P = \mathrm{supp}\ \varphi$ for some $0 \le \varphi \in M_\sigma(X)$ with $T^* \varphi = \varphi$,*

(ii) *N is the closure of the set $\{x : f(x) > 0\}$ for some $0 \le f \in C(X)$ with $\lim_n \|T_n f\| = 0$.*

Sketch of proof. Put $\alpha = \sup \{\mu(\mathrm{supp}\ \varphi) : 0 \le \varphi \in M_\sigma(X) \cap N_T^*\}$. As easily seen, there exists $0 \le \varphi \in M_\sigma(X) \cap N_T^*$ such that $\alpha = \mu(\mathrm{supp}\ \varphi)$. Define

$$P = \mathrm{supp}\ \varphi \quad \text{and} \quad N = X - P.$$

Then, since N is open, the family $\mathscr{F} = \{f \in C(X) : 0 \le f \le 1$ on X and $f = 0$ on $P\}$ satisfies

$$\sup \{f(x) : f \in \mathscr{F}\} = 1_N(x) \quad \text{for each} \quad x \in X,$$

where 1_N denotes the indicator function of N. Choose $f_n \in \mathscr{F} (n = 1, 2, \cdots)$ so that $f_n \le f_{n+1}$ for each $n \ge 1$ and

$$\lim_n \langle f_n, \mu \rangle = \sup \{ \langle f, \mu \rangle : f \in \mathscr{F} \}.$$

Put

$$g = \bigvee_{n=1}^{\infty} f_n .$$

Then we get $\langle g, \varphi \rangle = \lim_n \langle f_n, \varphi \rangle = 0$, and thus $g \in \mathscr{F}$. On the other hand, since μ is strictly positive, $g = 1$ on N. Hence P and N are both open and closed in X.

Let

$$\mathscr{E} = \{ f \in C(X) : f \geq 0 \text{ and } \lim_n \| T_n f \| = 0 \}.$$

Immediately, $f \in \mathscr{E}$ implies $\{x : f(x) > 0\} \subset N$; and there exists an $f \in \mathscr{E}$ such that

$$\mu(\{x : f(x) > 0\}) = \sup \{ \mu(\{x : g(x) > 0\}) : g \in \mathscr{E} \}.$$

We denote by A the closure of the set $\{x : f(x) > 0\}$ and show that $A = N$. Assume the contrary. Then A is a proper subset of N, and so there exists $0 \leq g \in C(X)$ such that

$$\emptyset \neq \{x : g(x) > 0\} \subset N - A .$$

Let λ be any weak*-limit point of the sequence $(T_n^* \mu)$. Since λ is invariant with respect to T^*, if λ_0 denotes the maximal σ-additive measure with $\lambda_0 \leq \lambda$, then $T^* \lambda_0 \leq \lambda_0$ and $T^*(\lambda - \lambda_0) \geq \lambda - \lambda_0$, as $T(\geq 0)$ is σ-additive. Hence $T^*(\lambda - \lambda_0) = \lambda - \lambda_0$ and $T^* \lambda = \lambda$, as $\|T\| \leq 1$. Since supp λ_0 is both open and closed in X, it follows that supp $\lambda_0 \subset P$, and thus

$$\langle g, \lambda_0 \rangle = 0 .$$

On the other hand, since μ and $\lambda - \lambda_0$ are disjoint in the sense of Ando [2], by Lemma 1 in [2] there exists an $h \in C(X)$, with $0 \leq h \leq g$, such that

$$\langle h, \mu \rangle \neq 0 \quad \text{and} \quad \langle h, \lambda - \lambda_0 \rangle = 0.$$

Then we obtain $\langle h, \lambda \rangle = \langle h, \lambda - \lambda_0 \rangle + \langle h, \lambda_0 \rangle = 0$, and

$$0 \leq \liminf_n \langle T^n h, \mu \rangle \leq \liminf_n \langle T_n h, \mu \rangle = 0.$$

Now, as in the proof of Theorem 2, we may and will assume that $T^* \mu$ is absolutely continuous with respect to μ. Then T^* can be regarded as a positive linear contraction operator on $L_1(\mu)$, and then modifying arguments in Foguel [5], pp. 40—43, there exists an $h' \in C(X)$, with $0 \leq h' \leq h$ and $h' \neq 0$, such that

$$\bigvee_{k=1}^{\infty} \left(\sum_{i=1}^{k} T^{n_i} h'\right) \leq 1$$

for some subsequence (n_i) of (n), and hence also such that

$$\lim_n \| T_n h' \| = 0.$$

But this is a contradiction, because $f + h' \in \mathscr{E}$ and $\mu(\{x : f(x) + h'(x) > 0\}) > \mu(\{x : f(x) > 0\})$. This completes the proof.

Proof of Theorem 3. (a) $\Longrightarrow$ (b): Suppose there exists a strictly positive σ-additive invariant measure μ with respect to T^*. Then, since T^* can be regarded as a positive linear operator on $L_1(\mu)$ with $T^*1 = 1$, we see by a mean ergodic theorem (cf. Theorem VIII. 5. 1 in [4]) that for any $\xi \in L_1(\mu)$ $T_n^* \xi$ converges in norm. Therefore, by Theorem 1 in [8], for any $f \in C(X)$ $\lim_n T_n f(x)$ exists almost everywhere with respect to μ. Thus, as in the proof of Theorem 2, we see that (O)-$\lim_n T_n f$ exists. It is immediate that if $0 \leq f \in C(X)$ and $f \neq 0$ then $\langle (\mathrm{O})\text{-}\lim_n T_n f, \mu \rangle = \langle f, \mu \rangle \neq 0$. Hence (b) follows.

(b) $\Longrightarrow$ (a): Suppose (b) holds. Let Y, Z and s be as in Lemma. Denote by $\hat{Y}$ the Stone-Čech compactification of Y. We define a positive linear operator S on $C(Y)$ by the relation

$$Sf(x) = s(x)^{-1} T(fs)(x) \qquad (f \in C(Y),\ x \in Y),$$

where $fs \in C(X)$ is defined by $(fs)(x) = f(x)s(x)$ if $x \in Y$ and $= 0$ if $x \in X - Y$. Since $C(\hat{Y})$ is identified with $C(Y)$, S can be regarded as an operator on $C(\hat{Y})$. Let us denote by $\hat{S}$ the operator on $C(\hat{Y})$ corresponding to S.

(I) *$\hat{Y}$ is quasi-Stonian.*

To see this, suppose $(\hat{f}_n)$ is a bounded sequence of continuous functions on $\hat{Y}$. Since $Y = \{x : s(x) > 0\}$ and X is 0-dimensional, there exists a disjoint countable family $\{Y_k\}$ of subsets of Y such that

(i) $Y = \bigcup_{k=1}^{\infty} Y_k$,

(ii) each Y_k is a subset of $\{x : s(x) \geq 1/k\}$ and both open and closed in X.

It follows that each Y_k is quasi-Stonian. Hence there exists an $f \in C(Y)$, with $f \leq \hat{f}_n$ on Y for each $n \geq 1$, such that $g \in C(Y)$ and

$g \leq \hat{f}_n$ on Y for each $n \geq 1$ imply $g \leq f$ on Y. Obviously, the continuous extension $\hat{f}$ of f to $\hat{Y}$ is the infimum of $(\hat{f}_n)$.

(II) *$\hat{S}$ is a σ-additive Markov operator on $C(\hat{Y})$.*

It is clear that $\hat{S}$ is a Markov operator on $C(\hat{Y})$. To see that $\hat{S}$ is σ-additive, let $\hat{h}_n \in C(\hat{Y})$ $(n = 1, 2, \cdots)$ be such that $\hat{h}_n \geq \hat{h}_{n+1} \geq 0$ for each $n \geq 1$ and $\bigwedge_{n=1}^{\infty} \hat{h}_n = 0$. Define $(h_n s)(x) = \hat{h}_n(x)s(x)$ if $x \in Y$ and $= 0$ if $x \in X - Y$. Then $h_n s \in C(X)$, $h_n s \geq h_{n+1} s \geq 0$ for each $n \geq 1$, and $\bigwedge_{n=1}^{\infty} h_n s = 0$ on X.

Since T is σ-additive, $\bigwedge_{n=1}^{\infty} T(h_n s) = 0$ on X, and hence

$$\bigwedge_{n=1}^{\infty} \hat{S}\hat{h}_n = \bigwedge_{n=1}^{\infty} (1/s)\, T(h_n s) = 0 \text{ on each } Y_k.$$

Therefore $\bigwedge_{n=1}^{\infty} \hat{S}\hat{h}_n = 0$ on $\hat{Y}$.

(III) *There exists a strictly positive σ-additive invariant measure in $M(\hat{Y})$ (with respect to $\hat{S}^*$).*

In fact, (b) implies that, by the definition of $\hat{S}$, (O)-$\limsup_n \hat{S}_n \hat{f} \neq 0$ for each $0 \leq \hat{f} \in C(\hat{Y})$ with $\hat{f} \neq 0$. Thus the proposition implies that, in order to prove (III), it is enough to observe that there exists a strictly positive σ-additive measure in $M(\hat{Y})$. For this purpose, let μ be a strictly positive σ-additive measure in $M(X)$ and ν its restriction to Y. Then ν can be regarded as a member of $M(\hat{Y})$, and it is a routine matter to see that, as a member of $M(\hat{Y})$, ν is a strictly positive and σ-additive.

(IV) *There exists a strictly positive σ-additive invariant measure in $M(X)$ (with respect to T^*).*

Let μ be as in (III), and let λ be a weak*-limit point of the sequence $(T_n^* \mu)$. Denote by λ_0 the maximal σ-additive measure with $\lambda_0 \leq \lambda$. We then have

$$T^* \lambda_0 \leq \lambda_0 ,$$

since $T^*\lambda = \lambda$. This, together with the fact that $Ts = s$ and $Y = \{x : s(x) > 0\}$, shows that

$$T^* \lambda_0 = \lambda_0 \quad \text{on} \quad Y.$$

We now show that $Y \subset \operatorname{supp} \lambda_0$. If this is not the case, take any $x \in Y - \operatorname{supp} \lambda_0$ and any nonnegative function $f \in C(X)$ such that $f = 0$ on $Z \cup \operatorname{supp} \lambda_0$ and $f(x) > 0$. Since μ and $\lambda - \lambda_0$ are disjoint in the sense of Ando [2], as in the proof of the proposition, there exists a function $g \in C(X)$, with $0 \leq g \leq f$, such that

$$\langle g, \mu \rangle > 0 \quad \text{and} \quad \langle g, \lambda \rangle = 0 .$$

On the other hand, (III) implies that $(\mathrm{O})\text{-}\limsup_n T_n(sg) = (\mathrm{O})\text{-}\liminf_n T_n(sg)$ on each Y_k, and hence on X, as $Y = \bigcup_{k=1}^{\infty} Y_k$ and $T_n(sg) = 0$ on $Z = X - Y$ for all $n \geq 1$. Furthermore, we have $(\mathrm{O})\text{-}\lim_n T_n(sg) \neq 0$. Since the σ-additivity of μ implies then that $\lim_n T_n(sg)(x) = [(\mathrm{O})\text{-}\lim_n T_n(sg)](x)$ for almost all $x \in X$ with respect to μ, it follows from Lebesgue's convergence theorem that

$$\langle sg, \lambda \rangle = \lim_n \langle T_n(sg), \mu \rangle = \langle (\mathrm{O})\text{-}\lim_n T_n(sg), \mu \rangle \neq 0 .$$

But this is impossible, because $|\langle sg, \lambda \rangle| \leq \|s\| \, |\langle g, \lambda \rangle| = 0$. Therefore we conclude that $Y \subset \operatorname{supp} \lambda_0$.

Define $\theta = \lim_n T_n^* \lambda_0$. Then $0 \leq \theta \in M_\sigma(X)$ and $T^* \theta = \theta$. Furthermore, we see that $\theta = \lambda_0$ on Y, and hence that $Y \subset \operatorname{supp} \theta$. To see that $X = \operatorname{supp} \theta$, let $0 \leq f \in C(X)$ be such that $\langle f, \theta \rangle = 0$. Then $T^n f = 0$ on supp θ for all $n \geq 0$, and thus

$$(\mathrm{O})\text{-}\limsup_n T_n f = 0 \quad \text{on} \quad Y \subset \operatorname{supp} \theta .$$

It follows that $(\mathrm{O})\text{-}\limsup_n T_n f = 0$ on X, since $(\mathrm{O})\text{-}\limsup_n T_n 1 \leq s$ and $s = 0$ on $Z = X - Y$. Hence (b) implies that $f = 0$ on X, and this proves that $X = \operatorname{supp} \theta$. The proof is completed.

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