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COMMUTATIVITY THEOREMS FOR CERTAIN RINGS

To Professor Gorô Azumaya on his sixtieth birthday

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Throughout the present paper, R will represent an associative ring with center C, and N the set of all nilpotent elements of R. Following [13], R is called a *left s-unital* (resp. *s-unital*) *ring* if for each $x \in R$ there holds $x \in Rx$ (resp. $x \in Rx \cap xR$). If R is a left s-unital (resp. s-unital) ring then for any finite subset F of R there exists an element e in R such that $ex = x$ (resp. $ex = xe = x$) for all $x \in F$ (see [13, Theorem 1] and [12, Lemma 1]). Such an element e will be called a *left pseudo-identity* (resp. *pseudo-identity*) of F. By [6, Proposition 2], the following conditions are equivalent:

a) For every $x \in R$ there exists an element x' in the subring generated by x and a positive integer m such that $x^m = x^{m-1}x'$.

b) For every $x \in R$ there exist positive integers m, k such that $x^m = x^{m+k}$.

c) For every $x \in R$ there exists a positive integer m such that $x^m = x^{2m}$.

If R satisfies one of the above equivalent conditions, following [6], we term R a *periodic ring*. Finally, R is said to be *normal* if every idempotent of R is in C.

The purpose of this paper is to prove the following commutativity theorems.

Theorem 1. *Let R be a left s-unital periodic ring, and n a fixed positive integer. Suppose that* i) *N is n-torsion free,* ii) $[x, [x, u]] = 0$ *for all $x \in R$ and $u \in N$,* iii) *$x - y \in N$ implies that $x^n = y^n$ or both x and y commute with all elements of N. Then R is a subdirect sum of local commutative rings.*

Theorem 2. *Let R be an s-unital periodic ring, and n a fixed positive integer. Suppose that* i) *N is n-torsion free,* ii) $[x, [x, u]] = 0$ *for all $x \in R$ and $u \in N$,* iii)′ *$x - y \in N$ implies that $x^n - y^n \in C$. If $(n, q^\alpha - 1) = 1$ for all prime factors q of n and for all positive integers α (especially, if n is a power of a prime), then R is a subdirect sum of local commutative rings.*

Theorem 3. *Let R be an s-unital ring, and n a fixed positive integer. If N is n-torsion free, then the following are equivalent:*

1) *R is commutative.*

2) *R satisfies the polynomial identities $[x^n, y^n]=0$ and $[x^k, [x^k, y^k]]=0$, where k is a positive integer with $(n, k)=1$.*

3) *R satisfies the polynomial identities $[x^n, y^n]=0$ and $[x^k, (xy)^k-(yx)^k]=0$, where k is a positive integer with $(n, k)=1$.*

4) *R satisfies the polynomial identities $(xy)^n-x^n y^n=0$ and $(xy)^{n+1}-x^{n+1}y^{n+1}=0$.*

5) *R satisfies the polynomial identity $(xy)^n-(yx)^n=0$.*

Theorem 1 is a slight generalization of [2, Theorem 3], and Theorem 2 is related to [1, Theorem]. The proof of these theorems will be given in § 1. Theorem 3 contains [4, Theorem 1] and all the results in [3], and § 2 is devoted to the proof of Theorem 3.

1. In advance of proving Theorems 1 and 2, we establish the following lemmas.

Lemma 1. (1) *Let n be a positive integer. If $[a, [a, b]]=0$, then $[a^n, b]=na^{n-1}[a, b]$.*

(2) *Suppose that* ii)′ *$a^2=0$ implies $a\in C$. Then R is normal.*

(3) *If* ii) *is satisfied, then R is normal.*

(4) *Let n be a positive integer. Suppose that $x-y\in N$ implies that $x^n-y^n\in C$ or $xy=yx$. Then $ux^n=x^n u$ for all $x\in R$ and $u\in N$, and necessarily R is normal.*

Proof. (1) and (2) are well known.

(3) Let e be an arbitrary idempotent of R. Given $x\in R$, we readily see that $(ex-exe)^2=0$. Hence, by ii), it follows that $ex-exe=[e, [ex-exe]]=0$, i. e. $ex=exe$. Similarly, we obtain $xe=exe$.

(4) It suffices to show that if $ux\neq xu$ then $ux^n=x^n u$. Since $(u+x)-x\in N$ and $(u+x)x\neq x(u+x)$, we have $c=(u+x)^n-x^n\in C$. Hence, $(u+x)x^n=(u+x)\{(u+x)^n-c\}=\{(u+x)^n-c\}(u+x)=x^n(u+x)$, which simplifies to $ux^n=x^n u$.

Lemma 2. *Let R be a left s-unital periodic ring, and n a fixed positive integer. If hypotheses* i) *and* iii) *are satisfied, then nilpotent elements of R commute with each other.*

Proof. Suppose that $uv\neq vu$ for some $u, v\in N$. Let e be a left pseudo-identity of $\{u, v\}$. Since R is periodic, there exists a positive integer m such that $e^m = e^{2m}$. Then, by Lemma 1 (4), $f = e^m$ is a central idempotent with $fu = u$ and $fv = v$. Obviously, $(u+f)v \neq v(u+f)$, and hence by hypothesis iii), $(u+f)^n = f^n$. Therefore, $u^n + \cdots + \binom{n}{2}u^2 + nu = 0$. For some $k \geq 2$, $u^k = 0$ and $u^{k-1} \neq 0$. Then

$$nu^{k-1} = \{u^n + \cdots + \binom{n}{2}u^2 + nu\}u^{k-2} = 0,$$

which is a contradiction. This contradiction shows that nilpotent elements of R commute with each other.

Lemma 3. *Let R be an s-unital periodic ring, and n a fixed positive integer. If hypotheses* i), ii) *and* iii)′ *are satisfied, then nilpotent elements of R commute with each other.*

Proof. Let $u, v \in N$, and e a pseudo-identity of $\{u, v\}$. By Lemma 1 (1) and (4), we can easily see that $n[u, v]v^{n-1} = 0$ and $n[u, v](e+v)^{n-1} = n[u, e+v](e+v)^{n-1} = 0$. Then

$$\begin{aligned} n[u, v]v^{n-2} &= n\{[u, v] + (n-1)[u, v]v + \cdots + [u, v]v^{n-1}\}v^{n-2} \\ &= n[u, v](e+v)^{n-1}v^{n-2} = 0. \end{aligned}$$

Repeating the same procedure, we obtain eventually $n[u, v] = 0$. From the proof of [11, Lemma], we can easily see that both uv and vu are nilpotent. Since $[uv, vu] = [uv, [v, u]] = u[v, [v, u]] + [u, [v, u]]v = 0$ by ii), $[u, v]$ is seen to be nilpotent. Hence, by i), $uv = vu$.

As was claimed in [9] (cf. also Lemma 1 (3)), a careful examination of the proof of [11, Theorem] shows that the following is still valid.

Lemma 4. *Suppose that* i)′ *nilpotent elements of R commute with each other. If R is a periodic ring satisfying* ii), *then R is a subdirect sum of local commutative rings and nil commutative rings.*

Finally, careful scrutiny of the proof of [1, Theorem] generalizes the part (b) of the theorem as follows :

Lemma 5. *Let R be a periodic ring, and n a fixed positive integer. If hypotheses* i)′ *and* iii)′ *are satisfied, and if $(n, q^\alpha - 1) = 1$ for all prime factors q of n and for all positive integers α, then R is a subdirect sum of local commutative rings and nil commutative rings.*

Proof of Theorems 1 *and* 2. Theorem 1 is an easy combination of

Lemmas 2 and 4. In fact, if R is a left s-unital periodic ring then hypotheses i) and iii) imply i)′ (Lemma 2), and hence R is a subdirect sum of local commutative rings (Lemma 4). Theorem 2 is a combination of Lemmas 3 and 5.

Remark. Let R be a normal periodic ring satisfying i)′. Then, from the proof of [11, Lemma], N is seen to be a commutative nil ideal, whence it follows that $[u, x]^2=0$ for all $x\in R$ and $u\in N$. In particular, if R is a periodic ring satisfying i)′ and ii)′, then the hypothesis ii) is satisfied. Combining this with Lemma 2, we see that in Lemma 4 and Theorem 1 the hypothesis ii) may be replaced by ii)′.

2. In preparation for the proof of Theorem 3, first we borrow several results of the previous paper [10], which are summarized in Lemmas 6 and 7 below (see [10, Lemma 1 and Theorem]).

Lemma 6. *Let m be a positive integer, and $a, b\in R$.*

(1) *If e is a pseudo-identity of $\{a, b\}$, then $a^m[a, b]=0$ and $(a+e)^m[a+e, b]=0$ imply $[a, b]=0$.*

(2) *Let R be an s-unital ring. If $x^m[x, b]=0$ for all $x\in R$, then b is in C.*

Lemma 7. *Let R be an s-unital ring. Then the following are equivalent:*

1) *R is commutative.*

2) *For each pair of elements x, y of R, there exist relatively prime, positive integers n and k such that $(xy)^n-(yx)^n=0$ and $(xy)^k-(yx)^k=0$.*

3) *There exist positive integers n, k with $(n, k)=1$ such that R satisfies the polynomial identities $[x^n, y^n]=0$ and $[x^k, y^k]=0$.*

Next, we state the following lemma which is evident by [5, Theorem 3], [7, Theorem 1] and [8, Theorem].

Lemma 8. *Let m, n be positive integers. If R satisfies one of the polynomial identities $(xy)^n-(yx)^n=0$, $(xy)^{n+1}-x^{n+1}y^{n+1}=0$ and $[x^n, y^m]=0$, then the commutator ideal $D(R)$ of R is contained in N (and N is an ideal of R).*

Corollary 1. *Let R be an s-unital ring, and let n, k be positive integers such that $(n, k)=1$. If $k>1$ and R satisfies the polynomial identities*

$(xy)^n-(yx)^n=0$ *and* $(x+y)^k-x^k-y^k=0$, *then R is commutative.*

Proof. Let x, y be arbitrary elements of R. Since $[x, y]$ is nilpotent by Lemma 8, there exists a positive integer α such that $[x, y]^{k^\alpha}=0$. Then, we can easily see that $(xy)^{k^\alpha}-(yx)^{k^\alpha}=0$. Hence, R is commutative by Lemma 7.

Corollary 2. *Let R be an s-unital ring, and let* $a, b\in R$. *Suppose that N is n-torsion free and R satisfies the polynomial identity* $[x^n, y^n]=0$. *If* $[a, [a, b]]=[b, [a, b]]=0$, *then* $[a, b]=0$.

Proof. By Lemma 8, N is an ideal containing $D(R)$. Let e be a pseudo-identity of $\{a, b\}$. By Lemma 1 (1) and the hypothesis that N is n-torsion free, we conclude $a^{n-1}[a, b^n]=0$. Similarly, $(a+e)^{n-1}[a+e, b^n]=0$. Then $[a, b^n]=0$ by Lemma 6 (1). Again by Lemma 1 (1) and the hypothesis that N is n-torsion free, we obtain $b^{n-1}[a, b]=0$. Now, repeating the above argument for $b+e$ instead of b, we can see that $(b+e)^{n-1}[a, b+e]=0$. Hence, $[a, b]=0$ by Lemma 6 (1).

The next lemma will be followed by an efficient corollary.

Lemma 9. *Let R be an s-unital ring, and* $b\in R$. *Suppose that* $D(R)$ *is nil and N is n-torsion free. If* $[x^n, b]=0$ *for all* $x\in R$, *then* $[u, b]=0$ *for all* $u\in N$.

Proof. Let e be a pseudo-identity of $\{u, b\}$. Since u is nilpotent, there exists a minimal positive integer m such that $[u^k, b]=0$ for all integers $k\geq m$. If $m\geq 2$, then

$$0=[(e+u^{nm-1})^n, b]=[e^n+nu^{m-1}+\cdots+u^{(m-1)n}, b]=n[u^{m-1}, b],$$

and hence $[u^{m-1}, b]=0$, which contradicts the minimality of m. Thus, $m=1$, and $[u, b]=0$.

Corollary 3. *Let R be an s-unital ring. Suppose that N is n-torsion free.*

(1) *If R satisfies the polynomial identity* $[x^n, y]=0$, *then R is commutative.*

(2) *If R satisfies the polynomial identity* $[x^n, y^n]=0$, *then* $[u, y^n]=0$ *for all* $y\in R$ *and* $u\in N$.

(3) *If R satisfies the polynomial identity* $[x^n, y^n]=0$, *then* $[u, v]=0$ *for all* $u, v\in N$.

Proof. (1) By Lemmas 8 and 9, $D(R)\subseteq N\subseteq C$. Hence, by Lemma 1 (1), it follows $nx^{n-1}[x, y]=[x^n, y]=0$. Since N is n-torsion free, we have $x^{n-1}[x, y]=0$, and hence by Lemma 6 (2), $[x, y]=0$.

(2) and (3) are immediate from Lemmas 8 and 9.

Finally, we shall prove the following

Lemma 10. *Let R be an s-unital ring. If R satisfies the polynomial identity $[x^n, y^n]=0$, then $k[x^n, y]=0$ for some positive integer k.*

Proof. Let x, y be arbitrary elements of R, and e a pseudo-identity of $\{x, y\}$. Then $[x^n, (y+e)^n]=0$ together with $[x^n, y^n]=0$ implies

$$[x^n, ny]+[x^n, \binom{n}{2}y^2]+\cdots+[x^n, ny^{n-1}]=0.$$

Replacing y by iy in this identity, we obtain

$$i[x^n, ny]+i^2[x^n, \binom{n}{2}y^2]+\cdots+i^{n-1}[x^n, ny^{n-1}]=0 \quad (i=1, 2, \cdots, n-1).$$

Hence, $nd[x^n, y]=d[x^n, ny]=0$, where $d(\neq 0)$ is the determinant of the matrix of integer coefficients in the above equations.

We are now in a position to complete the proof of Theorem 3.

Proof of Theorem 3. Obviously, 1) implies 2) — 5).

2) implies 1). By Corollary 2, R satisfies the polynomial identity $[x^k, y^k]=0$. Hence, it follows that R is commutative, by Lemma 7.

3) implies 1). By Corollary 3 (3), N is commutative. Moreover, $D(R)\subseteq N$ by Lemma 8, and hence N is a commutative ideal. Now, it is a routine to verify that $N^2\subseteq C$. Let $x\in R$, $u\in N$, and e a pseudo-identity of $\{x, u\}$. In view of $N^2\subseteq C$, we can easily see that

$$\{x(u+e)\}^k=(xu+x)^k\equiv x^k+x^ku+xux^{k-1}+\cdots+x^{k-1}ux \pmod{C}$$
$$\{(u+e)x^k\}=(ux+x)^k\equiv x^k+ux^k+xux^{k-1}+\cdots+x^{k-1}ux \pmod{C},$$

and therefore $[x^k, u]\equiv\{x(u+e)\}^k-\{(u+e)x\}^k \pmod{C}$. Since $[x^k, \{x(u+e)\}^k-\{(u+e)x\}^k]=0$, it follows that $[x^k, [x^k, u]]=0=[x^k+e, [x^k+e, u]]$. By Corollary 3 (2), $[x^{kn}, u]=0$ and $[(x^k+e)^n, u]=0$. Hence, by Lemma 1 (1) and Lemma 6 (1), we conclude that $[x^k, u]=0$ for all $x\in R$ and $u\in N$. Especially, recalling that $D(R)\subseteq N$, we see that for any x', y' in the subring R' generated by all k-th powers of elements of R the commutator $[x', y']$ is in the center of R'. Then, by Corollary 2, R' is commutative, namely R satisfies the polynomial identity $[x^k, y^k]=0$. Thus, R is commutative by Lemma 7.

4) implies 1). By Lemma 8, N is an ideal containing $D(R)$. Let x, y be arbitrary elements of R, and e a pseudo-identity of $\{x, y\}$. Then

$$x[x^n, y]y^n = x^{n+1}y^{n+1} - xyx^ny^n = x^{n+1}y^{n+1} - xy(xy)^n = 0,$$

and similarly $x[x^n, y+e](y+e)^n = 0$. Hence,

$$x[x^n, y]y^{n-1} = x[x^n, y+e](y+e)^ny^{n-1} = 0.$$

Repeating this argument, we obtain eventually $x[x^n, y] = 0$. If $x^m = 0$ then

$$\begin{aligned}0 &= \{e - x + x^2 - \cdots + (-x)^{m-1}\}(e+x)[(e+x)^n, y] = [(e+x)^n, y] \\ &= n[x, y] + [x_1, y]\end{aligned}$$

where $x_1 = \binom{n}{2}x^2 + \cdots + x^n$. As an immediate consequence, we see that $x^2 = 0$ implies $[x, y] = 0$. Now, by induction method, we assume that every nilpotent element of index at most $m-1$ is central. Then, according to $x_1^{m-1} = 0$, we readily obtain $0 = n[x, y]$, and therefore $[x, y] = 0$. Thus, we have proved that $N \subseteq C$. Hence, $[x, y] \in C$ for all $x, y \in R$. Combining this with $x[x^n, y] = 0$, by Lemma 1 (1) we obtain $0 = x[x^n, y] = nx^n[x, y]$. We have therefore seen that R satisfies the polynomial identity $x^n[x, y] = 0$. So, R is commutative by Lemma 6 (2).

5) implies 1). Let $a \in R$, $u \in N$, and e a pseudo-identity of $\{a, u\}$. If u_0 is the quasi-inverse of u, then $eu_0 = u_0e = u_0$ and the map $\sigma: R \to R$ defined by $x \to x - u_0x - xu + u_0xu$ is a ring automorphism of R. By hypothesis,

$$\begin{aligned}a^n &= \{(e-u)(e-u_0)a\}^n = \{(e-u_0)a(e-u)\}^n \\ &= \sigma(a)^n = \sigma(a^n) = (e-u_0)a^n(e-u),\end{aligned}$$

whence it follows $[u, a^n] = 0$. Now, let R^* be the subring generated by all n-th powers of elements of R. Then, by the above, the set N^* of nilpotent elements of R^* is contained in the center C^* of R^*. Moreover, by Lemma 8, $D(R^*)$ is nil and thus R^*/N^* is commutative. Let $x^*, y^* \in R^*$. Then $x^{*n}y^{*n} - (x^*y^*)^n \in N^* \subseteq C^*$, and hence by hypothesis

$$\begin{aligned}x^{*n}[x^*, y^{*n}] &= [x^*, x^{*n}y^{*n}] = [x^*, (x^*y^*)^n] = x^*(x^*y^*)^n - (x^*y^*)^n x^* \\ &= x^*(x^*y^*)^n - x^*(y^*x^*)^n = 0.\end{aligned}$$

Since R^* is obviously s-unital, Lemma 6 (2) implies that $[x^*, y^{*n}] = 0$ for all $x^*, y^* \in R^*$. Then, $[x^*, y^*] = 0$ by Corollary 3 (1). Thus, $[x^n, y^n] = 0$ for all $x, y \in R$. Choose a positive integer k such that $k[x^n, y] = 0$ (Lemma 10). As was shown at the opening, $[x^n, u] = 0$ for all $u \in N$. Since $[x^n, y] \in N$ (Lemma 8), we obtain therefore $[x^{nk}, y] = kx^{n(k-1)}[x^n, y] = 0$, by Lemma 1 (1). Hence, $x^{n^2k}y^n = (x^{nk}y)^n = (x^{nk-1}yx)^n = x^{n^2k-1}y^nx$, namely

$x^{n^2k-1}[x, y^n]=0$. Thus, by Lemma 6 (2), it follows $[x, y^n]=0$, and therefore $R=C$ again by Corollary 3 (1).

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Added in proof. In case R is a ring with identity, H. E. Bell [Math. Japonica 24 (1979), 473-478] has proved that if R is n-torsion free and satisfies the polynomial identity $(xy)^n-(yx)^n=0$ then R is commutative.