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EQUATIONAL DEFINABILITY OF ADDITION IN RINGS SATISFYING POLYNOMIAL IDENTITIES

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Boolean rings and Boolean algebras, though historically and conceptually different, were shown by Stone [3] to be equationally interdefinable. Indeed, in a Boolean ring, addition is definable in terms of multiplication and the successor operation (Boolean complementation) $x^\wedge = 1+x\ (=1-x)$. In Theorem 1 of [2] it was shown that this type of equational definability of addition also holds for rings satisfying the identity $x^m = x^n$ $(m \neq n)$ in which the idempotents are in the center. More generally, in Theorem 2 of [2] it was shown that this equational definability of addition still holds when the identity $x^m = x^n$ above is replaced by the identity $x^n = x^{n+1}f(x)$, $f(t) \in \boldsymbol{Z}[t]$. Our present objective is to show that Theorems 1 and 2 of [2] are equivalent, and to give elegant proofs of them which do not use the axiom of choice. We also give some new conditions which are shown to be equivalent to those in Theorems 1 and 2 of [2].

We begin with the following lemma, which is immediate from the proof of [1, Proposition 2]. However, for the sake of self-containedness, we shall give the proof.

Lemma. *If R is a ring with identity, then the following are equivalent*:

a) *There exists a positive integer n and a polynomial $f(t) \in \boldsymbol{Z}[t]$ such that $x^n = x^{n+1}f(x)$ for all $x \in R$.*

b) *There exists a positive integer m such that $|\langle x\rangle| \leq m$ for all $x \in R$, where $\langle x\rangle$ denotes the subring generated by x.*

c) *There exist positive integers m, k such that $x^m = x^{m+k}$ for all $x \in R$.*

Proof. c) $\Longrightarrow$ a). Trivial.

a) $\Longrightarrow$ b). Clearly $qR = 0$, where $q = |2^{n+1}f(2) - 2^n|\ (>1)$. We set $d = \deg f(t)\ (\geq 0)$, and $k = q^{d+1}$. If x is nilpotent, then we readily obtain $|\langle x\rangle| \leq q^n \leq q^{nk}$. Next, we consider the case that x is not nilpotent. Evidently, $e = x^n f(x)^n$ is an idempotent with $x^n = x^n e = (xe)^n$. Let $y = xe = ex$. Since $e = y^n f(y)^n$ and $y^n = y^{n+1}f(y)$, $y^* = f(y)e$ is the inverse of y in eRe. Then, it is easy to see that $|\langle y^*\rangle| \leq k$, and that $y^{*l} = e$ with some positive integer $l < k$. Hence, we obtain $(x^n)^l = (y^n)^l = (y^l)^n = e$, and thus $x^{nk} = x^{n(k-l)}$. Now, we readily obtain $|\langle x\rangle| \leq q^{nk}$.

b) $\Longrightarrow$ c). Let x be an arbitrary element of R. Since $|\langle x\rangle| \leq m$, we can find a positive integer $l \leq m$ such that $x^m = x^{m+l}$. Obviously, $x^m = x^{m+m!}$.

A ring R is said to be *normal* if every idempotent of R is central. As is well known, an idempotent of R is central if it commutes with all idempotents, or equivalently, with all nilpotents. We see therefore that the conditions 1) and 5) in the following proposition are equivalent to those assumed in [2, Theorem 2] and [2, Theorem 1], respectively.

Proposition. *If R is a ring with identity, then the following are equivalent*:

1) *R is normal and there exists a positive integer n and a polynomial $f(t) \in Z[t]$ such that $x^n = x^{n+1}f(x)$ for all $x \in R$.*

2) *There exists a positive integer n and a polynomial $f(t) \in Z[t]$ such that $x^n = x^{n+1}f(x)$ and $(xy)^n f(xy)^n = (yx)^n f(yx)^n$* for all *$x, y \in R$.*

3) *There exists a positive integer n and a polynomial $f(t) \in Z[t]$ such that $(xy)^n = (yx)^{n+1}f(yx)$ for all $x, y \in R$.*

4) *There exists a positive integer n and a polynomial $f(t) \in Z[t]$ such that $x^n = x^{n+1}f(x)$ and $(xy)^n = (yx)^n$ for all $x, y \in R$.*

5) *R is normal and there exist positive integers n, k such that $x^n = x^{n+k}$ for all $x \in R$.*

Proof. The equivalence of 3) and 4) is immediate, and that of 1) and 5) is included in Lemma.

5) $\Longrightarrow$ 4). It is easy to see that there exists a positive integer m such that $x^m = x^{2m} = x^{m+1} \cdot x^{m-1}$ for all $x \in R$. Now, let x, y be arbitrary elements of R. Since $(xy)^m$ and $(yx)^m$ are central idempotents, we have $(xy)^m = x(yx)^{m-1}(yx)^m y = (yx)^m(xy)^m$, and similarly $(yx)^m = (xy)^m(yx)^m$. Hence, $(xy)^m = (yx)^m$.

4) $\Longrightarrow$ 2). Actually, $(xy)^n f(xy)^n = f(xy)^n (yx)^{2n} f(yx)^n = (xy)^{2n} f(xy)^n f(yx)^n = (xy)^n f(yx)^n = (yx)^n f(yx)^n$.

2) $\Longrightarrow$ 1). Let e be an arbitrary idempotent of R. By 2), for any unit u of R we have $e = e^n = e^{2n}f(e)^n = e^n f(e)^n = (euu^{-1})^n f(euu^{-1})^n = (u^{-1}eu)^n f(u^{-1}eu)^n = u^{-1}(e^n f(e)^n)u = u^{-1}eu$. Hence, e commutes with all units, and therefore with all nilpotents. This proves that e is in the center.

Now, we are in a position to give an elegant proof of our main theorem which contains Theorems 1 and 2 of [2].

Theorem. *Let R be a ring with identity which satisfies any* (*and* *hence all) of the five equivalent conditions in the above Proposition. Then the "+" of R is equationally definable in terms of the " · " of R and the (unary) successor operation "^".*

Proof. Assume that the condition 5) in Proposition holds. Since $q = 2^{n+k} - 2^n$ is zero in R, we have $x^\vee = x - 1 = x + (q-1)$ for any $x \in R$. Let x, y be arbitrary elements of R. By hypothesis, $e = x^{nk}$ and $e' = (x+1)^{nk}$ are central idempotents with $ex^n = x^n$ and $e'(x+1)^n = (x+1)^n$. Without loss of generality, we may assume that n is odd. Since $1-e = (x^n+1)(1-e) = (x+1)^n(x^{n-1} - x^{n-2} + \cdots - x + 1)^n(1-e) = e'(1-e)$, we can easily see that

$$\begin{aligned} x+y &= (ey+x)e + \{e'(y-1)+x+1\}(1-e) \\ &= \{(ey+x)e+1\}[\{e'(y-1)+x+1\}(1-e)+1]-1 \\ &= \{x^{nk+1}(x^{nk-1}y+1)+1\}[(x+1)(x^{nk}-1)^2\{(x+1)^{nk-1}(y-1)+1\}+1]-1 \\ &= [\{x^{nk+1}(x^{nk-1}y)^\wedge\}^\wedge\{x^\wedge((x^{nk})^\vee)^2((x^\wedge)^{nk-1}y^\vee)^\wedge\}^\wedge]^\vee. \end{aligned}$$

This completes the proof, since $x^\vee = x + (q-1) = (\cdots((x^\wedge)^\wedge)^\wedge\cdots)^\wedge$, $q-1$ iterations.

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