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NOTE ON COMMUTATIVITY OF RINGS. II

Dedicated to Professor Gorô Azumaya on his sixtieth birthday

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Throughout the present note, R will represent an associative ring, N the set of natural numbers, and Z the set of integers. The purpose of this note is to present a theorem which includes H. Bell [1, Theorem 2] as well as M. Hongan and I. Mogami [2, Theorem].

Given an element x of R, we consider the following properties concerning x:

(P_2) For each $y \in R$ there exist $m, n \in N$ such that

$$(xy)^\alpha = x^\alpha y^\alpha, \qquad \alpha = m, m+1;$$
$$(yx)^\beta = y^\beta x^\beta, \qquad \beta = n, n+1.$$

(P_1) For each $y \in R$ there exist $m, m' \in N$ with $(m, m') = 1$ and $n \in N$ such that

$$(xy)^\alpha = x^\alpha y^\alpha, \qquad \alpha = m, m+1, m', m'+1;$$
$$(yx)^\beta = y^\beta x^\beta, \qquad \beta = n, n+1.$$

(Q_2) For each $y \in R$ there exists $m \in N$ such that

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Among the properties (P), (Q), (Q′) considered in [2] and those above, there holds the following: (P) $\Longrightarrow$ (P_1) $\Longrightarrow$ (P_2), (Q) $\Longrightarrow$ (Q_1) $\Longrightarrow$ (Q_2) and (Q′) $\Longrightarrow$ (Q_1') $\Longrightarrow$ (Q_2').

Let $m, k \in N$. Following [1], R is called an (m, k)-*ring* if it satisfies $(xy)^\alpha = x^\alpha y^\alpha$ for all integers α with $m \leqq \alpha \leqq m+k-1$. If m and n are relatively prime positive integers, then every element of a ring which is both an $(m, 2)$-ring and an $(n, 2)$-ring possesses the property (P_1).

Now, we can state our theorem as follows :

Theorem. *If every element of an s-unital ring R possesses one of the properties* (P_1), (Q_1), (Q_1'), *then R is commutative.*

The following are immediate consequences of our theorem.

Corollary 1 ([1, Theorem 2]). *If m and n are relatively prime positive integers, then any s-unital ring which is both an $(m, 2)$-ring and an $(n, 2)$-ring is commutative.*

Corollary 2 ([2, Theorem] and [3, Corollary 2 (a)]). *If each element of an s-unital ring R possesses one of the properties* (P), (Q), (Q′), *then R is commmtative.*

As in [2], careful scrutiny of the proof of [3, Corollary 2 (a)] shows that our theorem is an easy consequence of the next

Proposition (cf. [2, Proposition]). *If an element x of an s-unital ring R possesses one of the properties* (P_1), (Q_1), (Q_1'), *then for each $y \in R$ there exists an $r \in N$ such that $x^r[x, y]=0=[x, y]x^r$, where $[x, y]=xy-yx$.*

Moreover, [3, Corollary 1 (c)] can be generalized as follows :

Corollary 3. *Assume that an element x of an s-unital ring R possesses one of the properties* (P_1), (Q_1), (Q_1'). *If, for every $e \in R$ with $ex=xe=x$, $x+e$ possesses one of the properties* (P_1), (Q_1), (Q_1'), *then x is central.*

In advance of proving the proposition, we state two lemmas.

Lemma 1. *Assume that an element x of an s-unital ring R possesses one of the properties* (P_2), (Q_2), (Q_2'). *Let $y \in R$, and $r \in N$. If $x^r y=0$* (or $yx^r=0$), *then for each $z \in R \cup Z$ there exists $p \in N$ such that $x^p zy=0=yzx^p$.*

Proof. It suffices to show that if $xy=0$ (resp. $yx=0$) then $yx^p=0$ (resp. $x^p y=0$) with some $p \in N$. But, careful examination of the proofs of [3, Lemma 2 (a)] and [2, Lemma] shows that the last is still valid.

Lemma 2. *Let R be an s-unital ring. Let $x, y \in R$, and $m \in N$.*

(a) *If x possesses the property* (P_2) *and $x^m[x, y^m]y=0$, then for each*

$t \in N \cup \{0\}$ *there exists a* $p \in N$ *such that* $[x, y^{tm}]yx^p = 0$.

(b) *If* x *possesses the property* (Q_2) (*resp.* (Q_2')) *and* $[x, y^{m+1}]x^m = 0$ (*resp.* $x^m[x, y^{m+1}] = 0$), *then for each* $t \in N \cup \{0\}$ *there exists a* $p \in N$ *such that* $[x, y^{t(m+1)}]x^p = 0$.

Proof. We understand $xy^0 = x = y^0x$, and proceed with induction on t.

(a) Assume that $[x, y^{tm}]yx^h = 0$ with some $h \in N$. By Lemma 1, there exists $s \in N$ such that $[x, y^m]y^{tm+1}x^s = 0$. If $p = \max\{h, s\}$, then $[x, y^{(t+1)m}]yx^p = [x, y^m]y^{tm+1}x^p + y^m[x, y^{tm}]yx^p = 0$, which completes the induction.

(b) Assume that $[x, y^{t(m+1)}]x^h = 0$ with some $h \in N$. Again by Lemma 1, there exists $s \in N$ such that $[x, y^{m+1}]y^{t(m+1)}x^s = 0$. If $p = \max\{h, s\}$, then $[x, y^{(t+1)(m+1)}]x^p = [x, y^{m+1}]y^{t(m+1)}x^p + y^{m+1}[x, y^{t(m+1)}]x^p = 0$, completing the induction.

We are now ready to complete the proof of our proposition.

Proof of Proposition. We consider first the case that x possesses the property (P_1). There exist $m, m' \in N$ with $(m, m') = 1$ such that

$$(xy)^\alpha = x^\alpha y^\alpha, \qquad \alpha = m, m+1, m', m'+1.$$

As is easily seen, there holds that

$$x^m[x, y^m]y = 0, \qquad x^{m'}[x, y^{m'}]y = 0.$$

Without loss of generality, we may assume that $tm - t'm' = 1$ with some $t, t' \in N$. By Lemma 2(a), there exist then some $p, p' \in N$ such that

$$[x, y^{tm}]yx^p = 0, \qquad [x, y^{t'm'}]yx^{p'} = 0.$$

We set $p'' = \max\{p, p'\}$. Then

$$[x, y]y^{tm}x^{p''} = [x, y]y^{t'm'}yx^{p''} = [x, y^{tm}]yx^{p''} - y[x, y^{t'm'}]yx^{p''} = 0.$$

Hence, by Lemma 1, $x^q[x, y]y^{tm} = 0$ with some $q \in N$.

By [3, Lemma 1 (a)], we can find $e \in R$ such that $ex = xe = x$ and $ey = ye = y$. Repeating the above argument for $y+e$ instead of y, we readily see that $x^{q'}[x, y](y+e)^u = 0$ with some $q'(\geqq q)$, $u \in N$. If $tm > 0$, then $x^{q'}[x, y]y^{tm-1} = x^{q'}[x, y](y+e)^u y^{tm-1} = 0$. Continuing the same procedure (if necessary), we obtain eventually $x^{q'}[x, y] = 0$. Again applying Lemma 1, we see that $x^r[x, y] = 0 = [x, y]x^r$ with some $r \in N$.

Next, we consider the case that x possesses the property (Q_1) (resp. (Q_1')). There exist $m, m' \in N$ with $(m+1, m'+1) = 1$ such that

$$(xy)^\alpha = y^\alpha x^\alpha \text{ (resp. } (yx)^\alpha = x^\alpha y^\alpha), \quad \alpha = m, m+1, m', m'+1.$$

As is easily seen, there holds that

$[x, y^{m+1}]x^m=0,\ [x, y^{m'+1}]x^{m'}=0$ (resp. $x^m[x, y^{m+1}]=0,\ x^{m'}[x, y^{m'+1}]=0$).

Without loss of generality, we may assume here that $t(m+1)-t'(m'+1)=1$ with some $t, t' \in N$. By Lemma 2 (b), there exist then some $p, p' \in N$ such that

$$[x, y^{t(m+1)}]x^p=0, \qquad [x, y^{t'(m'+1)}]x^{p'}=0.$$

We set $p''=\max\{p, p'\}$. Then

$$\begin{aligned}[x, y]y^{t(m+1)}x^{p''} &= [x, y]y^{t'(m'+1)}yx^{p''} \\ &= [x, y^{t(m+1)}]yx^{p''}-y[x, y^{t'(m'+1)}]yx^{p''}=0.\end{aligned}$$

Hence, by Lemma 1, $x^q[x, y]y^{t(m+1)}=0$ with some $q \in N$. Now, by making use of the same argument as in the latter half of the first case, we can easily see the conclusion.

Remerk. Let R be the non-commutative ring considered in [2, Remark]. Then it is easy to see that

$$\begin{aligned}(xy)^\alpha &= x^\alpha y^\alpha, & \alpha &= 3, 4, 6, 7\,; \\ (xy)^\beta &= y^\beta x^\beta, & \beta &= 2, 3, 5, 6\end{aligned}$$

for all $x, y \in R$. This example shows that our theorem need not be true if the condition $(m, m')=1$ in (P_1) (resp. $(m+1, m'+1)=1$ in (Q_1) and (Q_1')) is replaced by $(m+1, m'+1)=1$ (resp. $(m, m')=1$).

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