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ON FULLY RIGHT IDEMPOTENT RINGS AND DIRECT SUMS OF SIMPLE RINGS

Dedicated to Professor Gorô Azumaya on his sixtieth birthday

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A ring R is said to be *fully right idempotent* if every right ideal of R is idempotent, or equivalently, if $a \in (aR)^2$ for any $a \in R$. Following [10], R is called a *right s-unital ring* if for each $x \in R$ there exists an element e such that $xe=x$. If $x_1, \cdots, x_n$ are arbitrary elements of a right s-unital ring R, then there exists $e \in R$ such that $x_i e = x_i$ for all x_i ([10, Theorem 1]). It is immediate that R is a fully right idempotent ring if and only if every non-zero ideal of R is a right s-unital ring.

In § 1, we shall prove that if R is a fully right idempotent ring with identity, G is a locally finite group which acts on R and the order of each element of G is a unit in R, then the skew group ring $R*G$ is also fully right idemotent (Theorem 1). As a particular case, Theorem 1 provides another proof for the "if" part of [3, Theorem 9]. We shall prove also that if R is a fully right idempotent ring and G is a finite group of automorphisms of R such that $|G|^{-1} \in R$, then the fixed subring R^G is fully right idempotent. In § 2, we shall give necessary and sufficient conditions for a ring to be a finite direct sum of simple rings with identity (Theorem 2). Then, [1, Theorem 3. 1], [9, Lemma 3. 1] and [5, Corollary 16] are corollaries of this theorem. Finally, we shall show that the group ring $R[G]$ is a finite direct sum of simple rings with identity if and only if R is a finite direct sum of simple rings with identity and G is a finite group such that $|G|^{-1} \in R$ (Theorem 3).

Throughout, R will represent a ring, $J(R)$ the Jacobson radical of R, and $\omega(G)$ the augmentation ideal of the group ring $R[G]$. For a subset I of R, $r(I)$ will denote the right annihilator of I in R.

1. Let G be a group which acts on R (by means of a homomorphism into the automorphism group of R). For $r \in R$ and $g \in G$ we will let r^g denote the image of r under g. The *skew group ring* $R*G$ is defined to be $\bigoplus_{g \in G} Rg$ with addition given component wise and multiplication given as follows: if $r, s \in R$ and $g, h \in G$, then $(rg)(sh) = rs^g gh$. If $x = \sum_{g \in G} r_g g$ is an element of $R*G$, then the support of x is the set Supp $(x) = \{g \in G \mid r_g \neq 0\}$.

Lemma 1. *Let G be a group which acts on R. If R is a fully right idempotent ring with identity, then $R*G/I$ is a flat left R-module for every ideal I of $R*G$.*

Proof. By [2, Corollary 11.23, p. 433], it suffices to show that $a\cdot R*G\cap I\subseteq aI$ for every $a\in R$. By induction with respect to n, we shall show that if $a(r_1g_1+\cdots+r_ng_n)\in I$, $r_i\in R$, $g_i\in G$, then $a(r_1g_1+\cdots+r_ng_n)\in aI$. Since R is fully right idempotent, $ar_1=ar_1\sum_{i=1}^{m} b_iar_1c_i$ with some b_i, $c_i\in R$, and therefore $ar_1g_1=ar_1\sum_{i=1}^{m} b_iar_1c_ig_1=ar_1\sum_{i=1}^{m} b_i(ar_1g_1)c_i^{g_1^{-1}}\in aI$, which proves the case $n=1$. Now, assume that $n>1$. As above, there exist a_i, $b_i\in R$ such that $ar_n=ar_n\sum_{i=1}^{m} b_iar_nc_i$. If we set $y=r_n\sum_{i=1}^{m} b_ia(r_1g_1+\cdots+r_ng_n)c_i^{g_n^{-1}}\in I$, we see that $v=a(r_1g_1+\cdots+r_ng_n-y)\in I$ and the cardinality of Supp (v) is less than n. By induction hypothesis, there exists then some $z\in I$ such that $v=az$. It follows therefore that $a(r_1g_1+\cdots+r_ng_n)=a(y+z)\in aI$.

We are now in a position to state our first theorem.

Theorem 1. *Let R be a fully right idempotent ring with identity, and G a locally finite group which acts on R. If the order of each element in G is a unit in R, then $R*G$ is fully right idempotent.*

Proof. We begin with proving the theorem for G of finite order. For each prime divisor p of $|G|$ there exists an element of G whose order is p, and therefore $|G|$ is a unit by assumtion. By [10, Proposition 5 (1)] and [2, Corollary 11.2, p. 433], $S=R*G$ is fully right idempotent if and only if S/I is a flat left S-module for each ideal I of S. By Lemma 1, S/I is a flat left R-module. Hence for any $a\in I$, there exists an R-homomorphism $\theta: S\to I$ such that $\theta(ga)=ga$ for all $g\in G$ (see [2, Proposition 11.27, p. 435]). As is easily verified, the map $\hat{\theta}: S\to I$ defined by $\hat{\theta}(s)=|G|^{-1}\sum_{g\in G} g^{-1}\theta(gs)$ is an S-homomorphism with $\hat{\theta}(a)=a$. Hence, ${}_SS/I$ is flat again by [2, Proposition 11.27]. Consequently, S is fully right idempotent.

Now, let G be a locally finite group, and x an arbitrary element of $R*G$. Since Supp (x) generates a finite subgroup H of G, we can apply the first step to see that $x\in(x\cdot R*H)^2\subseteq(x\cdot R*G)^2$. Thus, we have seen that $R*G$ is fully right idempotent.

Corollary 1 (see [3, Theorem 9 and Addendum]). *Let R be a ring with identity, and G a group. Then the group ring $R[G]$ is fully right* *idempotent if and only if* (a) *R is fully right idempotent,* (b) *G is locally finite, and* (c) *the order of each element in G is a unit in R.*

Proof. If (a), (b) and (c) hold, then $R[G]$ is fully right idempotent by Theorem 1. Conversely, if $R[G]$ is fully right idempotent, then $R \simeq R[G]/\omega(G)$ is fully right idempotent, and (b) and (c) hold by [9, Lemma 6. 5].

We shall conclude this section with the following :

Corollary 2. *Let R be a fully right idempotent ring with identity, and G a finite group of automorphitms of R such that $|G|^{-1} \in R$. Then the fixed subring $R^G = \{r \in R \mid r^g = r$ for all $g \in G\}$ is fully right idempotent.*

Proof. $R * G$ is fully right idempotent by Theorem 1, and $e = |G|^{-1}\sum_{g\in G} g$ is an idempotent of $R * G$. Since $R^G \simeq e(R * G)e$ by [4, Lemma 1. 2] and the proof of [4, Corollary 1. 4], it is obvious that R^G is fully right idempotent.

2. A ring R is said to have *the finite intersection property on right annihilators* provided that whenever $r(A)=0$ for a right ideal A of R there exists a finite subset F of A such that $r(F)=0$ (see [11]). As is easily seen, R possesses the property if and only if for any ideal A of R with $r(A)=0$, there exists a finite subset F of A with $r(F)=0$. It is also easy to see that every ring with minimum condition on right annihilators possesses the property.

A ring R (possibly without identity) is called a *right strongly semiprime ring* provided if I is an ideal of R and is essential as a right ideal then there exists a finite subset F of I with $r(F)=0$. A right strongly semiprime ring is semiprime (see [5]). As is easily seen, if R is a semiprime ring, then an ideal I of R is essential as a right ideal if and only if $r(I)=0$. Therefore we see that a ring R is a right strongly semiprime ring if and only if R is a semiprime ring and possesses the finite intersection property on right annihilators. D. Handelman [5, Corollary 16] (see also [7, Corollary 2. 8]) proved that any regular, right strongly semiprime ring with identity is a finite direct sum of simple rings.

Now, we shall prove the following :

Theorem 2. *The following conditions are equivalent :*

1) *R is a finite direct sum of simple rings with identity.*
2) *R is a right strongly semiprime, fully right idempotent ring.*

3) *R is a fully right idempotent ring and possesses the finite intersection property on right annihilators.*

Proof. By the above, 2) implies 3) and conversely.

1) $\Longrightarrow$ 2). It is clear that R is a right strongly semiprime ring. In order to see that R is fully right idempotent, it suffices to show that every simple ring with identity is fully right idempotent. In fact, if S is a simple ring with identity and I is a right ideal of S, then $I^2=(IS)I=I(SI)=IS=I$.

3) $\Longrightarrow$ 1). Let I be an arbitrary ideal of R, and choose an ideal K of R which is maximal with respect to the property that $I\cap K=0$. We set $L=I\oplus K$. Since R is semiprime and $(L\cap r(L))^2=0$, $r(L)$ has to be 0 by the choice of K. Hence, there exists a finite subset F of L with $r(F)=0$. Since the ideal S generated by F is a right s-unital ring, there exists an $e\in S$ such that $xe=x$ for all $x\in F$ ([10, Theorem 1]). Since $a-ea\in r(F)=0$ for all $a\in R$, e is a left identity of R. Now, let b be an arbitrary element of R, and choose an element f such that $(be-b)f=be-b$. Since $(be-b)f=bef-bf=bf-bf=0$, we obtain $be=b$, which means that e is the identity of R. Recalling here that e belongs to L, we readily obtain $R=L=I\oplus K$. We have therefore seen that R is a finite direct sum of simple rings with identity.

Combining Theorem 2 with [7, Theorem 3. 4], we can improve [7, Corollary 3. 5] as follows:

Corollary 3. *Let R be a ring with identity. Then the following are equivalent:*

1) *R is a direct sum of simple rings.*

2) *R is a fully right idempotent ring and every nonsingular quasi-injective right R-module is injective.*

3) *R is a fully right idempotent ring and every finite direct sum of nonsingular quasi-injective right R-modules is quasi-injective.*

4) *R is a fully right idempotent ring and every direct product of non-singular quasi-injective right R-modules is quasi-injective.*

As another application of Theorem 2, we shall present the following:

Corollary 4. *Let R be a fully right idempotent subring of a ring T. If T or $T/J(T)$ satisfies the descending chain condition on right annihilators, then R is a finite direct sum of simple rings with identity.*

Proof. First, we claim that $R\cap J(T)=0$. Let $z\in R\cap J(T)$, and choose $y\in RzR(\subseteq J(T))$ such that $z=zy$. Since $\{yx-x \mid x\in T\}=T$, it follows that $zT=0$, namely $z=0$. Consequently, R may be regarded as a subring of $T/J(T)$. Hence, in either case, R satisfies the descending chain condition on right annihilators. In particular, R possesses the finite intersection property on right annihilators, and therefore R is a finite direct sum of simple rings with identity (Theorem 2).

Now, the next is an immediate consequence of Corollary 4.

Corollary 5. (cf. [1, Theorem 31] and [9, Lemma 3. 1]). *Every right or left Goldie, fully right idempotent ring is a finite direct sum of simple rings with identity.*

Next, we shall give necessary and sufficient conditions for the group ring $R[G]$ to be a finite direct sum of simple rings. In preparation for the proof of Theorcm 3 we establish the following lemma.

Lemma 2. *Let R be a finite direct sum of simple rings with identity, and G a finite group which acts on R. If $|G|$ is a unit in R, then the skew group ring $R*G$ is a finite direct sum of simple rings.*

Proof. As is easily seen, $R*G$ is a completely reducible R-R-module. Let K be an arbitrary ideal of $R*G$. Then, K is a direct summand of ${}_RR*G$, and therefore by [4, Theorem 1. 3], K is a direct summand of ${}_{R*G}R*G$, say, $R*G=K\oplus L$ with some left ideal L of $R*G$. Recalling that $R*G$ is fully right idempotent (Theorem 1), we see that $K\cap L\cdot R*G=(K\cap L\cdot R*G)^2=KL\cdot R*G=0$, whence it follows that $R*G=K\oplus L\cdot R*G$. Thus, $R*G$ is a finite direct sum of simple rings.

Remark. By making use of Lemma 2 and the argument employed in the proof of Corollary 2, we can easily see that if R is a finite direct sum of simple rings with identity and G is a finite group of automorphisms of R such that $|G|^{-1}\in R$, then R^G is a finite direct sum of simple rings. This is a thcorem of Kharchenko [6] for R with identity.

Theorem 3. *Let R be a ring with identity, and G a group. Then the following are equivalent:*

1) *$R[G]$ is a finite direct sum of simple rings.*

2) *R is a finite direct sum of simple rings, and G is a finite group whose order is a unit in R.*

Proof. 2) $\Longrightarrow$ 1). This is included in Lemma 2.

1) $\Longrightarrow$ 2). Since $R[G]=\omega(G)\oplus I$ with some non-zero ideal I of $R[G]$,

we havc $r(\omega(G)) \neq 0$. Hence, G is a finite group (see [8, Lemma 2, p. 154]), and $|G|$ is a unit in R (Corollary 1). Finally, $R(\simeq R[G]/\omega(G)\simeq I)$ is obviously a finite direct sum of simple rings.

Corollary 6. *Let R be a ring with identity, and G a group. Then the following are equivalent:*

1) *$R[G]$ is a finite direct sum of simple, right Goldie rings.*

2) *R is a right Goldie, fully right (or left) idempotent ring and G is a finite group whose order is a unit in R.*

Proof. 1) $\Longrightarrow$ 2). Since $R[G]=\omega(G)\oplus I$ with some ideal I of $R[G]$, $R(\simeq I)$ is a right Goldie ring. The remaining is evident by Theorem 3.

2) $\Longrightarrow$ 1). Since R is a fiinite direct sum of simple rings (Corollary 5), $R[G]$ is also a finite direct sum of simple rings (Theorem 3). Noting that $R[G]_R$ is of finite Goldie dimension, we see that $R[G]_{R[G]}$ is of finite Goldie dimension. Combining this with the fact that the right singular ideal of $R[G]$ is zero, we readily see that $R[G]$ is a right Goldie ring.

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Added in proof. Let S be a separable extension of R. Then, it is easy to see that a left S-module M is flat whenever ${}_RM$ is flat. Now, assume further that R is fully right idempotent and ${}_RS$ has a free basis $\{s_1, \cdots, s_m\}$ such that $s_iR \supseteq Rs_i$ for all i. Then ${}_RS/I$ is flat for any ideal I of S (see the proof of Lemma 1), and therefore so is ${}_SS/I$, namely S is fully right idempotent. This proves the essential part in the proof of Theorem 1. Moreover, as another direct consequence of this fact, we see that if R is fully right idempotent then so is the full matrix ring $(R)_n$.

Finally, it should be mentioned that D. S. Passman and J. W. Fisher proved Lemma 2 back in 1977 (although never published). The author would like to thank Prof. J. W. Fisher for all the interest he has shown in the paper.