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ON AUTOMORPHISMS OF SKEW POLYNOMIAL RINGS OF DERIVATION TYPE

Dedicated to Prof. Gorô Azumaya on his 60th birthday

MIGUEL FERRERO and KAZUO KISHIMOTO

Throughout this paper, B will mean a ring with identity element 1. In [1], M. Rimmer established all B-ring automorphisms of a skew polynomial ring $B[X;\rho]$ $(=\sum_{i=0}^{\infty} X^iB)$ whose multiplication is given by $bX=X\rho(b)$ $(b\in B)$ where ρ is an automorphism of B, and he proved that for $Y=\sum_i X^ib_i \in B[X;\ \rho]$, the B-linear map $B[X;\ \rho]\to B[X;\ \rho]$ defined by $X^k\to Y^k$ is a B-ring automorphism if and only if $\rho^i(b)b_i=b_i\rho(b)$ $(b\in B)$, b_1 is invertible in B, and b_i is nilpotent for $i\geq 2$.

In this note, we shall deal with a skew polynomial ring $B[X;D]$ whose multiplication is given by $bX=Xb+D(b)$ where D is a derivation of B with $D(ab)=D(a)b+aD(b)$ $(a,\ b\in B)$. Our purpose now is to discuss conditions on $Y\in B[X;D]$ for the B-linear map $B[X;D]\to B[X;D]$ defined by $X^k\to Y^k$ to be a B-ring automorphism. The study starts with the preliminary section § 1, which contains several tool lemmas. §§ 2 and 3 contain our main results which are partially similar to those of Rimmer. In § 4, we shall deal mainly with a special case where B is torsion free.

1. In the rest of this note, N will mean the union of all nilpotent ideals of B, and A will mean a skew polynomial ring $B[X;D]$. Now, we shall begin our study with the following lemma.

Lemma 1. *Let $S=\{s_i\,|\,1\leq i\leq k\}$ be a set of nilpotent elements of B. If $s_iB\subset B_i=\sum_{p\geq i} Bs_p$ for all i, then $S\subset N$.*

Proof. Obviously, B_k is a nilpotent ideal. Let $\overline{B}=B/B_{i+1}$ (factor ring). Since $\overline{B}_i=\overline{Bs_i}$ is a nilpotent ideal of $\overline{B}$, by induction method we see that B_i is nilpotent. In particular, we have $S\subset N$.

Corollary 1. *Let $S=\{s_{ij}\ |1\leq i\leq h,\ 1\leq j\leq k\}$ be a set of nilpotent elements of B. If $s_{ij}B\subset \sum_{p\geq i,q\geq j} Bs_{pq}$ for all (i,j), then $S\subset N$.*

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Proof. Let N be the set of natural numbers. Then, as is well known, $N\times N$ has a linear order such that $(i, j)\geq(i', j')$ if 1) $i+j>i'+j'$ or 2) $i+j=i'+j'$ and $i\geq i'$. Since $\sum_{p\geq i,\, q\geq j} Bs_{pq}\subset\sum_{(p,q)\geq(i,j)} Bs_{pq}$, our assertion is immediate by Lemma 1.

Next, we shall make a remark on B-ring endomorphisms of A which plays an important rôle in the subsequent consideration. Let $Y=\sum_{i=0}^{n}X^{i}b_{i}\in A$ and b an arbitrary elemnt of B. Then

$$bY=\sum_{k=0}^{n} bX^{k}b_{k}=\sum_{k=0}^{n} (\sum_{i=0}^{k}\binom{k}{i}X^{i}D^{k-i}(b))b_{k}$$
$$Yb+D(b)=\sum_{i=0}^{n} X^{i}b_{i}b+D(b).$$

If the B-linear map ϕ : $A\rightarrow A$ defined by $X^{k}\rightarrow Y^{k}$ is a B-ring endomorphism then $bY=Yb+D(b)$ which implies the following

$$b_{i}b=\sum_{k=i}^{n}\binom{k}{i}D^{k-i}(b)b_{k}\qquad(i\geq 1)\tag{1.1}$$
$$b_{0}b+D(b)=\sum_{k=0}^{n} D^{k}(b)b_{k}.$$

Conversely, if the b_{i} satisfy the condition (1.1) then ϕ is a B-ring endomorphism.

Now, assume (1.1). Then, we have

$$\begin{aligned}D(b_{i})b&=D(b_{i}b)-b_{i}D(b)\qquad(i\geq 1)\\&=D(\sum_{k=i}^{n}\binom{k}{i}D^{k-i}(b)b_{k})-\sum_{k=i}^{n}\binom{k}{i}D^{k+1-i}(b)b_{k}\\&=\sum_{k=i}^{n}\binom{k}{i}D^{k-i}(b)D(b_{k})\end{aligned}$$

and hence, by induction method, we obtain

$$D^{r}(b_{i})b=\sum_{k=i}^{n}\binom{k}{i}D^{k-i}(b)D^{r}(b_{k})\qquad(i\geq 1,\ r\geq 0).\tag{1.2}$$

Obviously, b_{n} is a central element of B.

Moreover, if, for $Z=\sum_{j=0}^{m} X^{j}c_{j}\in A$, the B-linear map $A\rightarrow A$ defined by $X^{k}\rightarrow Z^{k}$ is a B-ring endomorphism then

$$D^{s}(c_{j})b=\sum_{h=j}^{m}\binom{h}{j}D^{h-j}(b)D^{s}(c_{h})\qquad(j\geq 1,\ s\geq 0)\tag{1.2$'$}$$

$$D^{r}(b_{i})D^{s}(c_{j})b=\sum_{k=i}^{n}\sum_{h=j}^{m}\binom{k}{i}\binom{h}{j}D^{k+h-i-j}(b)D^{r}(b_{k})D^{s}(c_{h})\tag{1.3}$$
$$(i\geq 1,\ j\geq 1,\ r\geq 0,\ s\geq 0).$$

2. In this section, we shall deal with B-ring automorphisms of B. The first study is the following

Lemma 2. *The B-linear map* $\phi: A \to A$ *defined by* $X^k \to (b_0 + Xb_1)^k$ *is a B-ring automorphism if and only if* b_1 *is a central unit and* $[b_0, b] = D(b)(b_1 - 1)$ *for all* $b \in B$.

Proof. Assume that ϕ is an automorphism and $\phi^{-1}(X) = \sum_{j=0}^m X^j c_j$. Then by (1. 1), b_1 is central in B and $[b_0, b] = b_0 b - b b_0 = D(b)(b_1 - 1)$ for all $b \in B$. Since $X = \phi^{-1}(b_0 + Xb_1) = b_0 + c_0 b_1 + Xc_1 b_1 + X^2 c_2 b_1 + \cdots + X^m c_m b_1$, the element b_1 is a unit. Conversely, assume that b_1 is a central unit and $[b_0, b] = D(b)(b_1 - 1)$ for all $b \in B$. Then, $[-b_0 b_1^{-1}, b] = D(b)(b_1^{-1} - 1)$ for all $b \in B$ and the B-ring endomorphism $\phi: A \to A$ defined by $X^k \to (-b_0 b_1^{-1} + Xb_1^{-1})^k$ is the inverse of ϕ.

In what follows, we assume always $D(N) \subset N$ and that the B-linear map $\phi: A \to A$ defined by $X^k \to (\sum_{i=0}^n X^i b_i)^k$ $(n \geq 2)$ is a B-ring automorphism. If $b_n \neq 0$, then $\phi^{-1}(X) = \sum_{j=0}^m X^j c_j$ with $c_m \neq 0$ and $m \geq 2$ (see Lemma 2).

Lemma 3. *If* $b_i c_j$ $(i \geq 1,\ j \geq 1)$ *are nilpotent provided* $i + j \geq h$, *then* $D^r(b_i) B c_j \subset N$ $(i + j \geq h,\ r \geq 0)$.

Proof. According to (1. 2), the assertion is equivalent to that $D^r(b_i)c_j \in N$. First, by (1. 3) and Cor. 1, $b_i c_j \in N$. Now, we shall proceed by induction on r. By induction hypothesis, $(D^{r+1}(b_i)c_j)^2 = (D(D^r(b_i)c_j) - D^r(b_i)D(c_j))D^{r+1}(b_i)c_j \in N$. Hence, by (1. 3) and Cor. 1, $D^{r+1}(b_i)c_j \in N$, completing the induction.

Lemma 4. $b_i c_j (i \geq 1,\ j \geq 1)$ *are nilpotent provided* $i + j \geq 3$.

Proof. Since $X = \phi\phi^{-1}(X) = \sum_{j=0}^m (\sum_{i=0}^n X^i b_i)^j c_j$, $b_n^m c_m = 0$ as the coefficient of the highest degree term. Recalling that b_n is central, we see that $b_n c_m$ is nilpotent. Now, assume that $n + m > k \geq 3$ and we have shown that $b_i c_j$ are nilpotent provided $i + j \geq k + 1$. Let $p + q = k (p,\ q \geq 1)$, and consider $d = D^{j_1}(b_{i_1}) \cdots D^{j_h}(b_{i_h}) c_h b_p c_q$, where $i_1 + \cdots + i_h \geq pq$. If $i_t > p$ for some t, then $i_t + q \geq k + 1$, and therefore $d \in N$ by Lemma 3. Next, if $h > q$, then $h + p \geq k + 1$ and $c_h b_p \in N$ by (1. 2)$'$ and Lemma 3, and hence $d \in N$. Finally, if $i_1, \cdots, i_h \leq p$ and $h \leq q$, then $h = q$ and $i_1 = \cdots = i_h = p$. Now, in the expansion of $\sum_{j=0}^m (\sum_{i=0}^n X^i b_i)^j c_j b_p c_q$, we can write the coefficient of X^{pq} as a sum of elements of the type d, where in case $h = q$ and $i_1 = \cdots = i_h$, d equals to $b_p^q c_q b_p c_q$. Hence, by the above consideration, we obtain $0 = b_p^q c_q b_p c_q + d'$ with some $d' \in N$. Let $\bar{R} = R/N$. Then, by (1. 1) and Lemma 3, there holds $\bar{c}_q \bar{b}_p = \bar{b}_p \bar{c}_q$. Hence, $(\bar{b}_p \bar{c}_q)^{2q} = 0$, which means that $b_p c_q$ is nilpotent. This completes the proof.

Now, we are at the position to prove the following theorem which is one of our main results.

Theorem 1. (a) *b_1 is a unit.*

(b) *b_i are nilpotent for $i \geq 2$, and therefore $\{b_i \mid i \geq 2\} \subset N$.*

Proof. (a) By Lemmas 3 and 4, $D^r(b_i)Bc_j \subset N$, provided $i+j \geq 3$. Hence the coefficient of X in $\sum_{j=2}^{m}(\sum_{i=0}^{n} X^i b_i)^j c_j$ is contained in N. Since $X = \sum_{j=0}^{m}(\sum_{i=0}^{n} X^i b_i)^j c_j$, we obtain $1 = b_1 c_1 + d$ with some $d \in N$. Hence, $b_1 c_1 = 1 - d$ is a unit, and similarly $c_1 b_1$ is a unit. It follows then that b_1 is a unit.

(b) Again by Lemmas 3 and 4, $D^r(b_i)c_j \in N$ $(j \geq 1)$. According to (1. 2), we have then $\bar{b}_i \bar{c}_1 = \bar{c}_1 \bar{b}_i$ in $\bar{R} = R/N$. Hence, $0 = (\bar{b}_i \bar{c}_1)^t = \bar{b}_i^t \bar{c}_1^t = 0$ with some t. Since c_1 is a unit by (a), it follows that $\bar{b}_i^t = 0$, and therefore b_i is nilpotent. The final assertion is immediate by Lemma 1 and (1. 1).

3. In this section, we assume that $D(N) \subset N$ and the B-linear map $\phi: A \to A$ defined by $X^k \to (\sum_{i=0}^{n} X^i b_i)^k$ is a B-ring endomorphism, namely (1. 1) is fulfilled. We assume further that b_1 is a unit. Then, we can easily see that $A = B[Y; \rho, E]$, where $Y = (X - b_0)b_1^{-1}$, $\rho: b \to b_1 b b_1^{-1}$, and E is the $(\rho, 1)$-derivation defined by $b \to \sum_{i=1}^{n} D^i(b) b_i b_1^{-1}$. We set $d_i = b_i b_1^{-1}$ $(i \geq 1)$. Then $\phi(Y) = \phi(X)b_1^{-1} - b_0 b_1^{-1} = \sum_{i=1}^{n} X^i d_i$ where $d_1 = 1$. Now, by N_0, we denote the ideal generated by $\{D^r(b_i) \mid i \geq 2,\ r \geq 0\}$. Then, by (1. 2), we have $N_0 = \sum_{r=0}^{\infty} \sum_{i=2}^{n} BD^r(b_i)$. Hence $D(N_0) \subset N_0$ and whence $D^r(N_0) \subset N_0$ $(r \geq 0)$. If b_i are nilpotent for $i \geq 2$, then by (1. 1) and Lemma 1, we have $b_i \in N$, and therefore $N_0 \subset N$. Obviously, $D^r(d_i) \in N_0$ $(r \geq 0)$. Moreover, for any finite subset $\{s_j \mid t \geq j \geq 0\}$ of B, we have

$$\sum_{j=2}^{t} \phi(Y)^j s_j = \sum_{i=2}^{t} X^i s_i + \sum_{i=2}^{u} X^i (\sum_{j=2}^{t} d_{ij} s_j)$$

where $\quad d_{ij} \in N_0$ $(i \geq 2,\ j \geq 2)$.

First, we shall prove the following

Theorem 2. *If b_i are nilpotent for $i \geq 2$, then ϕ is a monomorphism.*

Proof. Let $\sum_{j=0}^{t} Y^j s_j \in B$ and $\phi(\sum_{j=0}^{t} Y^j s_j) = \sum_{j=0}^{t} \phi(Y)^j s_j = 0$. By the above we have $s_0 = s_1 = 0$ and $s_i + \sum_{j=2}^{t} d_{ij} s_j = 0$ for some $d_{ij} \in N$ $(i \geq 2)$. Noting that the matrix $\begin{pmatrix} 1+d_{22} & d_{23} & \cdots & d_{2t} \\ \cdots & \cdots & \cdots & \cdots \\ d_{t2} & d_{t3} & \cdots & 1+d_{tt} \end{pmatrix}$ in $(B)_{t-1}$ is invertible modulo

$(N)_{t-1}$, we readily see that the matrix is invertible in $(B)_{t-1}$, and therefore $s_2=\cdots=s_t=0$.

Next, we shall prove the following

Theorem 3. *If N_0 is nilpotent, then ϕ is a B-ring automorphism.*

Proof. According to Th. 2, it remains only to show that $\phi(A)=A$. As is easily seen, AN_0 is an ideal of A. Since $X\equiv\phi(Y)$ (mod AN_0), we have $\sum_{i=0}^{\infty} X^iB\equiv\sum_{i=0}^{\infty}\phi(Y)^iB$ (mod AN_0). This implies $A=\sum_{i=0}^{\infty}X^iB=\sum_{i=0}^{\infty}\phi(Y)^iB+AN_0$. Hence, it is known that $A=\sum_{i=0}^{\infty}\phi(Y)^iB$.

Combining Lemma 1 with Th. 3, we readily obtain

Corollary 2. *Assume that one of the following conditions is fulfilled:*

1) *B is Noetherian.*

2) *There exists a monic polynomial $f(t)$ with coefficients in B such that $f(D)(b_i)=0$ for $i\geq 2$.*

If b_i are nilpotent for $i\geq 2$, then ϕ is a B-ring automorphism.

4. This section is about rings B with $D(N)\subset N$. At first, we shall prove the next

Proposition 1. *If B is torsion free, then $D(N)\subset N$.*

Proof. Let I be an ideal of B with $I^n=0$. Obviously, $D(I)+I$ is an ideal of B. If $s_1, s_2, \cdots, s_n$ are arbitrary elements of I, then $0=D^n(s_1s_2\cdots s_n)=n!D(s_1)D(s_2)\cdots D(s_n)-s$ with some $s\in I$, and hence we see that $n!D(I)^n\subset I$. Since $(n!)^nD(I)^{n^2}=0$, we obtain $D(I)^{n^2}=0$, and therefore $(D(I)+I)^{n^3}=0$. This proves the proposition.

In the rest of this note, ϕ will mean a B-linear map $A\to A$ defined by $X^k\to(\sum_{i=0}^{n}X^ib_i)^k$ where $b_i\in B$ $(i\geq 0)$. Now, we shall prove the following

Theorem 4. *Assume that B is torsion free. Assume further that* (1. 1) *is fulfilled, b_1 is a unit, and that b_i are central nilpotent elements for $i\geq 2$. Then ϕ is a B-ring automorphism.*

Proof. Since $D(C)\subset C$ (C the center of B), by (1. 2) we have $0=D^s(b_{n-1})D^{r-1}(b_i)-D^{r-1}(b_i)D^s(b_{n-1})=nD^r(b_i)D^s(b_n)$ $(r\geq 1,\ s\geq 0,\ i\geq 2)$. It follows therefore that $D^r(b_i)D^s(b_n)=0$. Next, if $n-1\geq 2$, then again by (1. 2) we have $0=D^s(b_{n-2})D^{r-1}(b_i)-D^{r-1}(b_i)D^s(b_{n-2})=\binom{n-1}{n-2}D^r(b_i)D^s(b_{n-1})+$

$\binom{n}{n-2} D^{r+1}(b_i) D^s(b_n) = (n-1) D^r(b_i) D^s(b_{n-1})$, and therefore $D^r(b_i) D^s(b_{n-1}) = 0$. Now, by an easy induction, we can see that $D^r(b_i) D^s(b_j) = 0$ ($r \geq 1$, $s \geq 0$, $i \geq 2$, $j \geq 2$). Hence, $N_0^2 = \sum_{i,j=2}^n b_i b_j B$ is nilpotent. Since N_0 is nilpotent as well, ϕ is a B-ring automorphism by Th. 3.

Remark 1. Assume that B is a commutative ring which is torsion free. Then, by Ths. 1 and 4, we see that ϕ is a B-ring automorphism if and only if (1. 1) fulfilled, b_1 is a unit, and that b_i are nilpotent for $i \geq 2$.

We shall conclude our study with the following remark.

Remark 2. Assume that B is semiprime, i. e. $N=0$. Then, $D(N) \subset N$ necessarily. Lemma 2 and Th. 1 together show that ϕ is a B-ring automorphism if and only if $b_i = 0$ ($i \geq 2$), b_1 is a central unit, and $[b_0, b] = D(b)(b_1 - 1)$ for all $b \in B$.

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UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL.
UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL
AND SHINSHU UNIVERSITY

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