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ON QF-2 ALGEBRAS WITH COMMUTATIVE RADICALS

Dedicated to Professor Gorô Azumaya on his 60th birthday

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Group algebras (of finite groups over an algebraically closed field) with commutative radicals have been studied by a number of authors: D. A. R. Wallace [4, 5, 6], S. Koshitani [1] and K. Motose and Y. Ninomiya [2]. In particular, Wallace has given, in [6], a result which determines the structure of blocks of group algebras of this type. The most important part of his result may be stated in the following form: *Let A be a block of a group algebra of the type mentioned above. If the radical N of A is such that $N^2 \neq 0$, then A is a commutative completely primary algebra.*

The purpose of the present note is to generalize this result to the case of QF-2 algebras in the sense of R. M. Thrall [3], over an arbitrary field K.

Theorem. *Let A be a QF-2 algebra over a field K and let A be itself a block. Assume that the radical N of A is commutative and N^2 does not vanish. Then A is a completely primary almost symmetric algebra over K such that the residue class algebra A/N is a (commutative) field. Moreover, if the base field K is perfect, then A is a commutative completely primary symmetric algebra over K.*

Proof. We note that, since N is commutative, N^2 is contained in the center of A. Let us first consider the case that K is an arbitrary field. We begin by proving the following contention: Let e and f be two primitive idempotents. If $eN^2 (= N^2e) \neq 0$, $ef = fe = 0$ and either $eAf \neq 0$ or $fAe \neq 0$, then eA and fA are isomorphic (as right A-modules). To show this, let M denote the left annihilator of N. If $eM = 0$, then $eMf = 0$. On the other hand, assume that $eM \neq 0$. Since eM is the unique minimal right A-submodule of eA, we have $eM \subseteq eN^2$, hence $eMf \subseteq eN^2f = efN^2 = 0$. Thus in either case we have $eMf = 0$. Since $eNf \cdot N = e \cdot N \cdot fN = e \cdot fN \cdot N = 0$, we have $eNf \subseteq eMf$, and therefore $eNf = 0$. Similarly we have $fNe = 0$. The condition that either $eAf \neq 0$ or $fAe \neq 0$ implies now that eA and fA are isomorphic. From the assumption that A is itself a block, together with what we have proved above, it follows that the indecomposable direct summands of the right regular module A are all isomorphic each other. Therefore A is a full matrix ring over a completely primary algebra. But, by the commutativity of N it follows that A itself is a completely primary algebra. By the same fact we have $(ab-ba)N^2=0$ for all $a, b \in A$. Since the left annihilator $l(N^2)$ of N^2 is a proper ideal of A, A/N is a field. Now let t be the nilpotency index of N, and m a nonzero element of N^{t-1}. Then we see that $M=N^{t-1}=Am$. Here m is a central element of A and the mapping $a+N \rightarrow am (a \in A)$ is an isomorphism of A/N onto M (as left and right A-modules). Therefore A is an almost symmetric algebra.

Now let K be a perfect field. Then there exists a subalgebra L of A which is isomorphic to A/N (as an algebra over K). Thus A is a direct sum of L and N, as a K-space. Since M is isomorphic to A/N as a left L-module, we get $M=Lm$. Let α be a generating element of L over K (i. e. $L=K(\alpha)$), and let $f(x)$ be the defining polynomial of α over K. To prove that A is commutative, it suffices to show that the primitive element α commutes with any $x \in N$. First of all one verifies directly that $x\alpha-\alpha x \in M$; hence one can choose an element λ in L to write $x\alpha=\alpha x+\lambda m$. We can then establish, by induction, the formula $x\alpha^t=\alpha^t x+t\lambda\alpha^{t-1}m$ $(t=1, \cdots, \deg f(x))$. From this it follows that $0=xf(\alpha)$ $=f(\alpha)x+\lambda f'(\alpha)m=\lambda f'(\alpha)m$, and hence $\lambda=0$. This proves that A is a commutative symmetric algebra.

Example. If K is not perfect in Theorem, then A is not necessarily commutative. To show this let us construct an example.

Let F be a field of characteristic 2, $P=F(t)$ the field of rational functions in one variable t over F, and $K=F(t^2)$. For an arbitrary element $\alpha=a+bt$ $(a, b \in K)$ of P, let $\tilde{\alpha}$ denote b, the coefficient of t. Then $\widetilde{\alpha\beta}=\tilde{\alpha}\beta+\alpha\tilde{\beta}$ for any two elements α, β in P. Let A be an associative algebra over K defined in the following way :

1) A is a 3-dimensional left vector space over P with a basis $\{1, m, m^2\}$.

2) The multiplication in A is defined by the rule

$$m^3=0 \text{ and } m\alpha=\alpha m+\tilde{\alpha}m^2 \text{ for any } \alpha \in P.$$

Then A is a non-commutative almost symmetric algebra over K such that the radical $N=Pm+Pm^2$ is commutative.

Corollary. *Let A be a weakly symmetric algebra over a field K and let A be itself a block. Assume that the radical N of A is commutative.*

Then A is of one of the following three types:

(1) *A is a simple algebra over K.*

(2) *A is a full matrix ring over a completely primary weakly symmetric algebra B over K such that the square of the radical $N'(=N\cap B)$ of B vanishes. (In this case B/N' is a division algebra and N' is one-dimensional as a left B/N'-space as well as a right one.)*

(3) *A is a completely primary almost symmetric algebra over K such that A/N is a field.*

Proof. In view of Theorem, we have only to consider the case that $N\neq 0$ and $N^2=0$. Let e be an arbitrary primitive idempotent. Then Ne is isomorphic to Ae/Ne as a left A-module. Hence the indecomposable left ideal Ae has only one (non-isomorphic) composition factor. Noting that A is itself a block, we can see that A is a full matrix ring over a completely primary algebra B. It is now easy to see that B is an algebra as described in our corollary.

If, in the corollary, we assume moreover that K is perfect, we can say something more.

(i) When A is of type (2), B satisfies the following conditions:

a) B is a 2-dimensional left D-space with a basis $\{1, m\}$, where D is a finite dimensional division subalgebra of B over K.

b) The multiplication in B is given by the rule

$$m^2=0 \text{ and } m\alpha=\sigma(\alpha)m \text{ for any } \alpha\in D,$$

where σ is a K-algebra automorphism of D.
Conversely, let B be an associative algebra over K satisfying the conditions a) and b), and let A be a full matrix ring over B. Then A is a weakly symmetric algebra over K with radical of square zero.

(ii) When A is of type (3), then, by Theorem, A is a commutative completely primary symmetric algebra.

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