Math. J. Okayama Univ. 22 (1980), 5—15

COHOMOLOGY FOR COMODULES

Dedicated to Professor Gorô Azumaya on his 60th birthday

ATSUSHI NAKAJIMA

In [1], D. W. Jonah shows that the equivalence classes of singular coalgebra extensions are related to certain 2-cocycles. Here we show that the first cohomology modules are directly connected with extensions of comodules. This treatment is a dual continuation of D. S. Trushin's paper [4].

In § 1, we introduce the notion of cohomology modules for comodules M and N and discuss the first cohomology modules. § 2 begins with the definition of the relative cohomology modules of comodules with respect to a coalgebra map $\Phi : C \longrightarrow D$. An exact cohomology sequence is exhibited, and certain functorial properties of the sequence are examined. Moreover we give a necessary and sufficient conditions for a comodule to be a relative injective or a relative projective.

Throughout this paper, k is a fixed field. All vector spaces are k-vector spaces and all linear maps are k-linear. Unadorned $\otimes$, Hom mean $\otimes_k$, Hom_k, respectively. For a coalgebra C, C-comodules always mean left C-comodules.

1. Cohomology for comodules. A *coalgebra* is a triple (C, Δ, ε), where C is a vector space, $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \longrightarrow k$ are linear maps such that $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$ and $(\varepsilon \otimes 1)\Delta = 1 = (1 \otimes \varepsilon)\Delta$. If C and D are coalgebras, a *coalgebra map* $\Phi : C \longrightarrow D$ is a linear map such that $(\Phi \otimes \Phi)\Delta_C = \Delta_D \Phi$. A C-comodule is a pair (X, ρ), where X is a vector space and $\rho : X \longrightarrow C \otimes X$ is a linear map such that $(1 \otimes \rho)\rho = (\Delta \otimes 1)\rho$ and $(\varepsilon \otimes 1)\rho = 1$. A *C-comodule map* $f : X \longrightarrow Y$ of C-comodules is a linear map such that $\rho_Y f = (f \otimes 1)\rho_X$. For any vector space V, we set $V^n = V \otimes \cdots \otimes V$ (n-timees), and $V^0 = k$.

In case $\Phi : C \longrightarrow D$ is a coalgebra map, any C-comodule X is a D-comodule with respect to the structure map $(\Phi \otimes 1)\rho_X$.

Let M, N be C-comodules, and n a non-negative integer. For the vector space $\mathrm{Hom}(M, D^n \otimes N)$ and $\mathrm{Hom}(M, D^{n+1} \otimes N)$, we define a linear map

$$\partial_n : \mathrm{Hom}(M, D^n \otimes N) \longrightarrow \mathrm{Hom}(M, D^{n-1} \otimes N)$$

by

$$\delta_n(f) = (\Phi \otimes f)\rho_M + \sum_{i=1}^{n} (-1)^i (1_{i-1} \otimes \Delta \otimes 1_{n-i} \otimes 1)f + (-1)^{n+1}(1_n \otimes (\Phi \otimes 1)\rho_N)f,$$

where $1_j : D^j \longrightarrow D^j$ is the identity map. Since $\rho\,(=\rho_M$ or $\rho_N)$ is a comodule structure map, Φ is a coalgebra map and $(\Delta \otimes 1)(\Phi \otimes 1)\rho = (\Phi \otimes \Phi \otimes 1)(1 \otimes \rho)\rho$, we can easily check that

$$\delta_{n+1}\delta_n(f) = \sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^i(-1)^j(1_{i-1}\otimes\Delta\otimes 1_{n+1-i}\otimes 1)(1_{j-1}\otimes\Delta\otimes 1_{n-j}\otimes 1)f.$$

Moreover, by induction, we have

$$\sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^i(-1)^j(1_{i-1}\otimes\Delta\otimes 1_{n+1-i})(1_{j-1}\otimes\Delta\otimes 1_{n-j}) = 0$$

and so $\delta_{n+1}\delta_n = 0$. Thus we obtain the cochain complex

$$0 \longrightarrow \mathrm{Hom}(M, N) \xrightarrow{\delta_0} \mathrm{Hom}(M, D\otimes N) \xrightarrow{\delta_1} \cdots$$
$$\cdots \xrightarrow{\delta_{n-2}} \mathrm{Hom}(M, D^{n-1}\otimes N) \xrightarrow{\delta_{n-1}} \mathrm{Hom}(M, D^n\otimes N) \xrightarrow{\delta_n}$$
$$\rightarrow \mathrm{Hom}(M, D^{n+1}\otimes N) \longrightarrow \cdots .$$

In the rest of this section, we put always $C = D$ and $\Phi = 1$.

Definition 1.1. Given D-comodules M and N, we set $C^n(M, N : D) = \mathrm{Ker}(\delta_n)$ and $B^n(M, N : D) = \mathrm{Im}(\delta_{n-1})$. The vector space $H^n(M, N : D) = C^n(M, N : D)/B^n(M, N : D)$ is said to be the *n-th cohomology module of M into N with coefficients in D.*

Let M and N be D-comodules. A *D-comodule extension of M by N* is a short exact sequence $E = (\nu, \mu : K) : 0 \longrightarrow N \xrightarrow{\nu} K \xrightarrow{\mu} M \longrightarrow 0$ of D-comodules and D-comodule maps. Two D-comodule extensions $E = (\nu, \mu : K)$ and $E' = (\nu', \mu' : K')$ of M and N are *equivalent* if there exists a D-comodule maps $\beta : K \longrightarrow K'$ such that the following diagram is commutative

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{\nu} & K & \xrightarrow{\mu} & M & \longrightarrow & 0 \\
 & & \| \, 1_N & & \downarrow \beta & & \| \, 1_M & & \\
0 & \longrightarrow & N & \xrightarrow[\nu']{} & K' & \xrightarrow[\mu']{} & M & \longrightarrow & 0.
\end{array}$$

In case the short exact sequence $E = (\nu, \mu : K)$ splits, E is said to be a *split D-comodule extension of M by N.*

Given D-comodules M, N, and $f \in C^1(M, N : D)$, we construct a

D-comodule K_f as follows: $K_f = N \oplus M$ as vector space, and define a linear map

$$\rho : K_f = N \oplus M \longrightarrow (D \otimes N) \oplus (D \otimes M) = D \otimes (N \oplus M)$$

by $\rho(n, m) = (\rho_N(n) + f(m), \rho_M(m))$. Then it is easy to see that

$$(1 \otimes \rho)\rho(n, m) = ((1 \otimes \rho_N)\rho_N(n) + (1 \otimes \rho_N)f(m) + (1 \otimes f)\rho_M(m), (1 \otimes \rho_M)\rho_M(m)).$$

Moreover, since $\delta_1(f) = 0$, we have

$$(*) \qquad (1 \otimes f)\rho_M + (1 \otimes \rho_N)f = (\Delta \otimes 1)f.$$

Therefore

$$(1 \otimes \rho)\rho(n, m) = ((\Delta \otimes 1)(\rho_N + f)(n), (\Delta \otimes 1)\rho_M(m)) = (\Delta \otimes 1)\rho(n, m).$$

Applying $(1 \otimes \varepsilon \otimes 1)$ to $(*)$, we have $f = (1 \otimes (\varepsilon \otimes 1)f)\rho_M + f$ and so $(\varepsilon \otimes 1)f = 0$, because ρ_M is a monomorphism. Hence we have

$$(\varepsilon \otimes 1)\rho(n, m) = ((\varepsilon \otimes 1)\rho_N(n) + (\varepsilon \otimes 1)f(m), (\varepsilon \otimes 1)\rho_M(m)) = \rho(n, m).$$

We have therefore seen that K_f is a D-comodule.

Now, let $\iota : N \longrightarrow K_f$ be the injection map and let $\tau : K_f \longrightarrow M$ be the projection map. Then it is easy to see that $E = (\iota, \tau : K_f)$ is a D-comodule extension of M by N.

Theorem 1.2. *Let M and N be D-comodules. If $\bar{f} = \bar{g}$ in $H^1(M, N : D)$, then the two extensions $E = (\iota, \tau : K_f)$ and $E' = (\iota, \tau : K_g)$ are equivalent.*

Proof. Since $\bar{f} = \bar{g}$ in $H^1(M, N : D)$, there exists $\xi \in \mathrm{Hom}(M, N)$ such that $g - f = \delta_0 \xi = (1 \otimes \xi)\rho_M - \rho_N \xi$. Define a map $\beta : K_f \longrightarrow K_g$ by $\beta(n, m) = (n + \xi(m), m)$. Clearly, β is linear and bijective. Let ρ' be the D-comodule structure map of K_g. Then, by $g - f = (1 \otimes \xi)\rho_M - \rho_N \xi$, we have

$$\begin{aligned} \rho'\beta(n, m) &= (\rho_N(n) + \rho_N \xi(m) + g(m), \rho_M(m)) \\ &= (\rho_N(n) + (1 \otimes \xi)\rho_M(m) + f(m), \rho_M(m)) \\ &= (1 \otimes \beta)\rho(n, m). \end{aligned}$$

Therefore $K_f \cong K_g$ as D-comodule, and the equivalence of $E = (\iota, \tau : K_f)$ and $E' = (\iota, \tau : K_g)$ is easily seen.

According to Th. 1.2, we may denote any $E = (\iota, \tau : K_f)$ by $(M, N : \bar{f})$.

Theorem 1.3. *Let M and N be D-comodules. If $E=(\nu, \mu : K)$ i. a D-comodule extension of M by N, then there exists a unique element f in $H^1(M, N : D)$ such that $E=(\nu, \mu : K)$ is equivalent to $(M, N : \bar{f})$.*

Proof. Let $\sigma : M \longrightarrow K$ be an inverse linear map of μ. We define $f_\sigma : M \longrightarrow D \otimes K$ by $f_\sigma(m) = ((1 \otimes \sigma)\rho_M - \rho_K\sigma)(m)$. Then, by $\mu\sigma = 1$ and $(1 \otimes \mu)\rho_K = \rho_M\mu$, we have $(1 \otimes \mu)f_\sigma = \rho_M - \rho_M\mu\sigma = 0$ and thus

$$\operatorname{Im}(f_\sigma) \subseteq \operatorname{Ker}(1 \otimes \mu) = \operatorname{Im}(1 \otimes \nu) = D \otimes N.$$

Moreover, by $(1 \otimes \rho_M)\rho_M = (\Delta \otimes 1)\rho_K$, we obtain that

$$\begin{aligned}\delta_1(f_\sigma) &= (1 \otimes (1 \otimes \sigma)\rho_M - 1 \otimes \rho_K\sigma)\rho_M - (\Delta \otimes 1)((1 \otimes \sigma)\rho_M - \rho_K\sigma)\\ &\quad + (1 \otimes \rho_N)((1 \otimes \sigma)\rho_M - \rho_K\sigma)\\ &= -(1 \otimes \sigma_K)((1 \otimes \sigma)\rho_M - \rho_K\sigma) + (1 \otimes \rho_N)((1 \otimes \sigma)\rho_M - \rho_K\sigma = 0.\end{aligned}$$

Hence $f_\sigma \in C^1(M, N : D)$.

First, we claim that $\bar{f}_\sigma$ is independent of the choice of σ. If $\omega : M \longrightarrow K$ is another linear map such that $\mu\omega = 1$, then, by $\mu(\sigma - \omega) = 0$, we have

$$\delta_0(\sigma - \omega) = (1 \otimes \sigma - 1 \otimes \omega)\rho_M - (\rho_K\sigma - \rho_K\omega) = f_\sigma - f_\omega,$$

that is, $\bar{f}_\sigma = \bar{f}_\omega$.

Now we define a map $g : K \longrightarrow (M, N : \bar{f}_\sigma)$ by $g(k) = (\sigma\mu(k) - k, \sigma\mu(k))$. (We identify $\sigma(M)$ with M.) Clearly, g is bijective and linear. Let ρ_σ be the D-comodule structure map of $(M, N : \bar{f}_\sigma)$. Then, by $k - \sigma\mu(k) \in N$ and the definition of f_σ, we have

$$\begin{aligned}\rho_\sigma g(k) &= (\rho_N(\sigma\mu(k) - k) + f_\sigma(\mu(k)),\ \rho_M\mu(k))\\ &= (\rho_K\sigma\mu(k) - \rho_K(k) + f_\sigma(\mu(k)),\ \rho_M\mu(k))\\ &= ((1 \otimes \sigma\mu)\rho_K(k) - \rho_K(k),\ (1 \otimes \mu)\rho_K(k))\\ &= (1 \otimes g)\rho_K(k).\end{aligned}$$

Thus g is a D-comodule map. Since $\tau g(k) = \mu(k)$ and $g\nu(n) = -\nu(n)$, $E = (\nu, \mu : K)$ and $(M, N ; \bar{f}_\sigma)$ are equivalent.

Corollary 1.4. *A D-comodule extension K of M by N is split if and only if K corresponds to the 0 class of $H^1(M, N : D)$.*

Proof. In Th. 1.3, if $K = N \oplus M$ and if $\theta : K \longrightarrow M$ is the projection, then there exists a D-comodule map $\sigma : M \longrightarrow K$ such that $\sigma\theta = 1$.

Thus $f_\sigma = (1 \otimes \sigma)\rho_M - \rho_K\sigma = 0$, and so $\bar{f}_\sigma = \bar{0}$ in $H^1(M, N: D)$. The converse is immediate.

By Cor. 1. 4, we have the following

Corollary 1. 5. *Let M be a D-comodule. Then*

(1) *M is a projective D-comodule if and only if $H^1(M, N: D) = 0$ for every D-comodule N.*

(2) *M is an injective D-comodule if and only if $H^1(N, M: D) = 0$ for every D-comodule N.*

Corollary 1. 6. *A coalgebra D is cosemisimple* ([3, p. 290, Def.]) *if and only if $H^1(M, N: D) = 0$ for every pair of D-comodules M and N.*

2. **Relative cohomology.** Throughout this section, let $\Phi : C \longrightarrow D$ be a coalgebra map. If X is a C-comodule, then X is a D-comodule via $(\Phi \otimes 1)\rho_X$.

Proposition 2. 1. *Let M and N be C-comodules. Then for any non-negative integer n, there exists a linear map ϕ_n : $\mathrm{Hom}(M, C^n \otimes N) \longrightarrow \mathrm{Hom}(M, D^n \otimes N)$ such that the following diagram is commutative*

$$\begin{array}{ccc} \mathrm{Hom}(M,\ C^n \otimes N) & \xrightarrow{\delta_n} & \mathrm{Hom}(M,\ C^{n+1} \otimes N) \\ \phi_n \downarrow & & \downarrow \phi_{n+1} \\ \mathrm{Hom}(M,\ D^n \otimes N) & \xrightarrow[\delta_n]{} & \mathrm{Hom}(M,\ D^{n+1} \otimes N). \end{array}$$

Proof. We define ϕ_n by $\phi_n(f)(x) = (\Phi^n \otimes 1)f(x)$ $(x \in M)$. Then it is clear that ϕ_n is a linear map, and we have

$$\begin{aligned} \phi_{n+1}\delta_n(f) &= (\Phi^{n+1}\otimes 1)(1\otimes f)\rho_M + \textstyle\sum_{i=1}^{n}(-1)^i(\Phi^{n+1}\otimes 1)(1_{i-1}\otimes\Delta\otimes 1_{n-i}\otimes 1)f \\ &\quad + (-1)^{n+1}(\Phi^{n+1}\otimes 1)(1_n\otimes\rho_N)f \\ &= (\Phi\otimes(\Phi^n\otimes 1)f)\rho_M + \textstyle\sum_{i=1}^{n}(-1)^i(1_{i-1}\otimes\Delta\otimes 1_{n-i}\otimes 1)(\Phi^n\otimes 1)f \\ &\quad + (-1)^{n+1}(1_n\otimes\Phi\otimes 1)(\Phi^n\otimes\rho_M)f \\ &= \delta_n\phi_n(f). \end{aligned}$$

If $\Phi : C \longrightarrow D$ is an epimorphism, then there exists a linear map $\psi : D \longrightarrow C$ such that $\Phi\psi = 1$. Therefore, for any $g \in \mathrm{Hom}(M, D^n \otimes N)$, we can define a linear map $f: M \longrightarrow C^n \otimes N$ by $f = (\psi^n \otimes 1)g$, and then $\phi_n(f) = g$. Thus we have the following

Proposition 2.2. *If $\Phi : C \longrightarrow D$ is an epimorphism, then ϕ_n is an epimorphism for any non-negative integer n.*

Since $\phi_{n+1}\partial_n = \partial_n\phi_n$, ∂_n maps $\mathrm{Ker}(\phi_n)$ into $\mathrm{Ker}(\phi_{n+1})$. Therefore the following proposition is clear.

Proposition 2.3. *Let $\bar{\partial}_n$ be the restriction of ∂_n on* $\mathrm{Ker}(\phi_n)$. *Then the diagram*

$$\begin{array}{ccc} \mathrm{Ker}(\phi_n) & \xrightarrow{\iota_n} & \mathrm{Hom}(M, C^n \otimes N) \\ \bar{\partial}_n \downarrow & & \downarrow \partial_n \\ \mathrm{Ker}(\phi_{n+1}) & \xrightarrow[\iota_n]{} & \mathrm{Hom}(M, C^{n+1} \otimes N) \end{array}$$

is commutative, where ι_n is the canonical inclusion. Moreover we have $\bar{\partial}_{n+1}\bar{\partial}_n = 0$.

Definition 2.4. Let M and N be C-comodules. We set

$C^n(M, N: C, D) = \{f \in C^n(M, N: D) \mid \phi_n(f) = 0\} = \mathrm{Ker}(\bar{\partial}_n)$ and $B^n(M, N: C, D) = \{f \in B^n(M, N: C) \mid f = \partial_{n-1}(g)$ for some g with $\phi_{n-1}(g) = 0\} = \mathrm{Im}(\bar{\partial}_{n-1})$, and $H^n(M, N: C, D) = C^n(M, N: C, D)/B^n(M, N: C, D)$ is called *the n-th cohomology module of M into N with coefficients in C relative to D.*

Theorem 2.5. *Let $\Phi : C \longrightarrow D$ be an epimorphism and let M and N be C-comodules. Then there exists an exact sequence of vector spaces*

$$\cdots \to H^{n-1}(M, N: D) \xrightarrow{\bar{c}_n} H^n(M, N: C, D) \xrightarrow{\bar{\iota}_n} H^n(M, N: C) \xrightarrow{\bar{\phi}_n} H^n(M, N: D) \to \cdots.$$

Proof. For $\bar{f} \in H^n(M, N: C, D)$, we put $\bar{\iota}_n(\bar{f}) = \overline{\iota_n(f)}$. If $\bar{f} = 0$, then $f = \partial_{n-1}(f')$ For some $f' \in \mathrm{Ker}(\phi_{n-1})$. By $\partial_n\iota_n = \iota_{n+1}\partial_n$, we have

$$\bar{\iota}_n(\bar{f}) = \overline{\iota_n\partial_{n-1}(f')} = \overline{\partial_{n-1}\iota_n(f')} = \overline{\partial_{n-1}(f')} = 0.$$

Thus $\bar{\iota}_n$ is well defined, and linear. Similarly if we set $\bar{\phi}_n(\bar{g}) = \overline{\phi_n(g)}$, then, by $\partial_n\phi_n = \phi_{n-1}\partial_n$, $\bar{\phi}_n$ is well defined and linear.

Now, we define $\bar{c}_n$. Let $\bar{f} \in H^{n-1}(M, N: D)$ with $f \in C^{n-1}(M, N: C)$. Since Φ is an epimorphism, there exists $f' \in \mathrm{Hom}(M, C^{n-1} \otimes N)$ such that $\phi_{n-1}(f') = f$. Since

$$0 = \partial_{n-1}(f) = \partial_{n-1}\phi_{n-1}(f') = \phi_n\partial_{n-1}(f'),$$

we have $\partial_{n-1}(f') \in \mathrm{Ker}(\phi_n)$. If $f' - f'' \in \mathrm{Ker}(\phi_{n-1})$, then

$$0 = \partial_{n-1}\phi_{n-1}(f' - f'') = \phi_n\partial_{n-1}(f' - f''),$$

and so $\partial_{n-1}(f' - f'') \in \mathrm{Ker}(\phi_n)$, that is, $\partial_{n-1}(f' - f'') = 0$. This proves that $\partial_{n-1}(f')$ is independent of the choise of f'.

Suppose $\bar{f} = 0$ in $H^{n-1}(M, N : D)$. Then $f = \partial_{n-2}(g)$ for some $g \in \mathrm{Hom}(M, D^{n-2} \otimes N)$ and $\phi_{n-2}(g') = g$ for some $g' \in \mathrm{Hom}(M, C^{n-2} \otimes N)$. Since Φ is an epimorphism, there exists $f' \in \mathrm{Hom}(M, C^{n-1} \otimes N)$ such that $\phi_{n-1}(f') = f$. Then we have

$$\phi_{n-1}\partial_{n-2}(g') = \partial_{n-2}\phi_{n-2}(g') = \partial_{n-2}(g) = f = \phi_{n-1}(f'),$$

and so $\phi_{n-1}(f' - \partial_{n-2}(g')) = 0$. Therefore $\partial_{n-1}(f' - \partial_{n-2}(g')) = \partial_{n-1}(f')$ and $\bar{c}_n$ define by $\bar{c}_n(\bar{f}) = \overline{\partial_{n-1}(f')}$ is well defined.

Finally, we need to show the following ;

(1) $\mathrm{Im}(\bar{c}_n) = \mathrm{Ker}(\bar{\iota}_n)$,

(2) $\mathrm{Im}(\bar{\iota}_n) = \mathrm{Ker}(\bar{\phi}_n)$, and

(3) $\mathrm{Im}(\bar{\phi}_n) = \mathrm{Ker}(\bar{c}_{n+1})$.

First, we prove (1). If $\bar{f} \in \mathrm{Im}(\bar{c}^n)$, then $\bar{f} = \overline{\partial_{n-1}(g)}$ for some $g \in \mathrm{Hom}(M, C^{n-1} \otimes N)$. Then

$$\bar{\iota}_n(\bar{f}) = \bar{\iota}_n(\partial_{n-1}(g)) = \overline{\iota_n\partial_{n-1}(g)} = \overline{\partial_{n-1}(g)} = 0 \text{ in } H_n(M, N : C).$$

Thus $\mathrm{Im}(\bar{c}_n) \subseteq \mathrm{Ker}(\bar{\iota}_n)$. Conversely, if $\bar{\iota}_n(\bar{f}) = \overline{\iota_n(f)} = \bar{f} = 0$ in $H^n(M, N : C)$, then $f = \partial_{n-1}(g)$ for some $g \in \mathrm{Hom}(M, C^{n-1} \otimes N)$. Since $\phi_n(f) = 0$, we have

$$\partial_{n-1}\phi_{n-1}(g) = \phi_n\partial_{n-1}(g) = \phi_n(f) = 0.$$

Thus $\phi_{n-1}(g) \in C^n(M, N : C)$, and so $\bar{f} = \overline{\partial_{n-1}(g)} = \bar{c}_n\overline{(\phi_{n-1}(g))}$.

Next, we prove (2). Since $\phi_n\iota_n = \iota_{n+1}\phi_n$, $\mathrm{Im}(\bar{\iota}_n) \subseteq \mathrm{Ker}(\bar{\phi}_n)$ is clear. Let $\bar{f} \in \mathrm{Ker}(\bar{\phi}_n)$. Then $\phi_n(f) = \partial_{n-1}(g)$ for some $g \in \mathrm{Hom}(M, D^{n-1} \otimes N)$ and $\phi_{n-1}(g') = g$ for some $g' \in \mathrm{Hom}(M, C^{n-1} \otimes N)$. Since

$$\phi_n(f) = \partial_{n-1}\phi_{n-1}(g') = \phi_n\partial_{n-1}(g'),$$

we have $f - \partial_{n-1}(g') \in \mathrm{Ker}(\phi_n)$. Moreover, by $\partial_n(f - \partial_{n-1}(g')) = 0$, $f - \sigma_{n-1}(g') \in C^n(M, N : C, D)$. Therefore

$$\bar{\iota}_n\overline{(f - \partial_{n-1}(g'))} = \overline{f - \partial_{n-1}(g')} = \bar{f} - \overline{\partial_{n-1}(g')} = \bar{f},$$

that is, $\mathrm{Ker}(\bar{\phi}_n) \subseteq \mathrm{Im}(\bar{\iota}_n)$.

Finally we prove (3). Let $f \in \mathrm{Hom}(M, C^n \otimes N)$. Then, by the definition of $\bar{c}_n$, we have

$$\bar{c}_{n+1}\bar{\phi}_n(\bar{f}) = \bar{c}_{n+1}\overline{(\phi_n(f))} - \overline{\delta_{n-1}(f)} = 0 \text{ in } H^n(M, N: C, D),$$

and so $\mathrm{Im}(\bar{\phi}_n) \subseteq \mathrm{Ker}(\bar{c}_{n+1})$. Conversely, let $f \in \mathrm{Ker}(\bar{c}_n)$. Then $\phi_n(g) = f$ for some $g \in \mathrm{Hom}(M, C^n \otimes N)$, and $0 = \bar{c}_{n+1}(\bar{f}) = \overline{\delta_n(g)}$, that is,

$\delta_n(g) = \delta_n(g')$ for some $g' \in \mathrm{Hom}(M, C^n \otimes N)$ and $\phi_{n-1}(g') = 0$. Since $g - g' \in C^n(M, N: C)$, we have $\phi_{n-1}(g - g') = \phi_{n-1}(g) = f$. Thus $\bar{f} = \bar{\phi}_{n-1}(g - g')$. This shows that $\mathrm{Im}(\bar{\phi}_{n-1}) \subseteq \mathrm{Ker}(\bar{c}_n)$, completing the proof.

We denote the exact cohomology sequence in Th. 2.5 by $H(M, N: C, D, \Phi)$.

Definition 2.6. *Let* $V: \cdots \to V_n \to V_{n+1} \to V_{n+2} \to \cdots$ *and* $W: \cdots \to W_n \to W_{n-1} \to W_{n+2} \to \cdots$ be exact sequences of vector spaces, and $\Phi = \{\psi_i : V_i \to W_i \mid \psi_i \text{ is linear}\}$. We say that Ψ is a *morphism of exact sequences* if every square of the following diagram is commutative:

$$\begin{array}{ccccccccc} \cdots \to & V_n & \longrightarrow & V_{n+1} & \longrightarrow & V_{n+2} & \to \cdots \\ & \psi_n \downarrow & & \psi_{n+1} \downarrow & & \psi_{n+2} \downarrow & \\ \cdots \to & W_n & \longrightarrow & W_{n+1} & \longrightarrow & W_{n+2} & \to \cdots. \end{array}$$

We say that Ψ is *surjective* (resp. *injective, bijective, split*) if each ψ_i is surjective (resp. injective, bijective, split).

Theorem 2.7. *Let M, N, N' be C-comodules and let $\tau: N \to N'$ be a C-comodule map. Then there exists a covariant morphism $\Omega: H(M, N: C, D, \Phi) \to H(M, N': C, D, \Phi)$ and a contravariant morphism $\Omega': H(N, M: C, D, \Phi) \to H(N', M: C, D, \Phi)$.*

Proof. Consider the following diagram

$$\begin{array}{ccccccccc} \cdots \to & H^{n-1}(M, N: D) & \xrightarrow{\bar{c}_n} & H^n(M, N: C, D) & \xrightarrow{\bar{\iota}_n} & H^n(M, N: C) & \xrightarrow{\bar{\phi}_n} & H^n(M, N: D) & \to \cdots \\ & H^{n-1}\tau_D \downarrow & & H^n\tau_0 \downarrow & & H^n\tau_C \downarrow & & H^n\tau_D \downarrow & \\ \cdots \to & H^{n-1}(M, N': D) & \xrightarrow[\bar{c}_n]{} & H^n(M, N': C, D) & \xrightarrow[\bar{\iota}_n]{} & H^n(M, N': C) & \xrightarrow[\bar{\phi}_n]{} & H^n(M, N': D) & \to \cdots, \end{array}$$

where the maps $H^n\tau_0$, $H^n\tau_C$ and $H^n\tau_D$ are defined as follows: For any

$f \in C^n(M, N: C)$, we set

$$H^n\tau(\bar{f}) = \overline{(1_n \otimes \tau)f},$$

where $H^n\tau = H^n\tau_0$ or $H^n\tau_C$. $H^n\tau_D$ is define in the same way. If $f \in C^n(M, N: C)$, then

$$\begin{aligned}\delta_n(1_n\otimes\tau)f &= (1\otimes(1_n\otimes\tau)f)\rho_M + \textstyle\sum_{i=1}^n(-1)^i(1_{i-1}\otimes\Delta\otimes 1_{n-i}\otimes 1)(1_n\otimes\tau) f\rho_M \\ &\quad + (-1)^{n+1}(1_n\otimes\rho_N)(1_n\otimes\tau)f \\ &= (1_{n+1}\otimes\tau)(1\otimes f)\rho_M + \textstyle\sum_{i=1}^n(-1)^i(1_{n+1}\otimes\tau)(1_{i-1}\otimes\Delta\otimes 1_{n-i}\otimes 1)f\rho_M \\ &\quad + (-1)^{n-1}(1_{n+1}\otimes\tau)(1_n\otimes\rho_N)f \\ &= (1_{n+1}\otimes\tau)(\delta_n(f)) = 0.\end{aligned}$$

Thus $(1_n \otimes \tau)f \in C^n(M, N' : C)$. Moreover, if $f \in C^n(M, N: C, D)$, then

$$\phi_n(1_n \otimes \tau)f = (\phi_n \otimes 1)(1_n \otimes \tau)f = (1_n\otimes\tau)(\phi_n \otimes 1)f = (1_n \otimes \tau)\phi_n(f) = 0,$$

and so $(1_n\otimes\tau)f \in C^n(M, N: C, D)$. Now, we show that $H^n\tau_0$ is well defined. Note that f is in $B^n(M, N: C, D)$ if and only if $f = \delta_{n-1}(g)$ and $\phi_{n-1}(g) = 0$ for some $g \in \mathrm{Hom}(M, C^{n-1} \otimes N)$. If $f \in B^n(M, N: C, D)$, then

$$(1_n\otimes\tau)f = (1_n\otimes\tau)\delta_{n-1}(g) = \delta_{n-1}((1_{n-1}\otimes\tau)g)$$

and

$$\phi_{n-1}((1_{n-1}\otimes\tau)g) = (1_{n-1}\otimes\tau)\phi_{n-1}(g) = 0.$$

Thus $(1_n\otimes\tau)f \in B^n(M, N: C, D)$, that is, $H^n\tau_0$ is well defined. Similarly $H^n\tau_C$ and $H^n\tau_D$ can be seen to be well defined.

We need to show that the above diagram is commutative. For $\bar{f} \in H^n(M, N: C)$, we have

$$\begin{aligned}H^n\tau_D(\bar{\phi}_n(\bar{f})) &= H^n\tau_D\overline{(\phi_n(f))} = \overline{(1_n\otimes\tau)\phi_n(f)} = \overline{\phi_n(1_n\otimes\tau)f} \\ &= \bar{\phi}_n\overline{((1_n\otimes\tau f)} = \bar{\phi}_n(H^n\tau_C(\bar{f})).\end{aligned}$$

Hence $H^n\tau_D \cdot \bar{\phi}_n = \bar{\phi}_n \cdot H^n\tau_C \cdot$ For $\bar{g} \in H^n(M, N: C, D)$,

$$H^n\tau_C(\bar{\iota}_n(\bar{g})) = H^n\tau_C(\bar{g}) = \overline{1_n \otimes \tau)g} = \bar{\iota}_n(H^n\tau_0(\bar{g})),$$

whence $H^n\tau_C \cdot \bar{\iota}_n = \bar{\iota}_n \cdot H^n\tau_0$. Finally, let $\bar{h}$ be in $H^{n-1}(M, N: D)$. Then, by $\phi_{n-1}((1_{n-1}\otimes)\tau k) = (1_{n-1}\otimes\tau)h$, we have

$$\begin{aligned}H^n\tau_0(\bar{c}_n(\bar{h})) &= H^n\tau_0\overline{(\delta_{n-1}(k))} = \overline{(1_n\otimes\tau)\delta_{n-1}(k)} = \overline{\delta_{n-1}((1_{n-1}\otimes\tau)k)} \\ &= \bar{c}_n(H^n\tau_D(\bar{h})),\end{aligned}$$

where $k \in \mathrm{Hom}(M, C^{n-1} \otimes N)$ and $\phi_{n-1}(k) = h$, completing the proof.

Corollary 2.8. *Let M, N, C, D be as in* Th. 2.7. *If $\tau: N \longrightarrow N'$ is a split surjective (resp. injective) C-comodule map, then $\Omega: H(M, N: C, D, \Phi) \longrightarrow H(M, N': C, D, \Phi)$ is a split surjective (resp. injective).*

Corollary 2.9. *Let M, N, C, D be as above. If $N = \bigoplus_\alpha N_\alpha$ as C-comodule, then $H(M, N: C, D, \Phi) = \bigoplus_\alpha H(M, N_\alpha: C, D, \Phi)$ and $H(N, M: C, D, \Phi) = \Pi_\alpha H(N_\alpha, M: C, D, \Phi)$.*

Definition 2.10. A C-comodule M is said to be (C, D)-*projective* if any exact sequence of C-comodules $0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0$ splits whenever it splits as a sequence of D-comodules. Dually (C, D)-*injective* comodules are defined.

Theorem 2.11. (1) *A C-comodule M is (C, D)-projective if and only if $\bar{\zeta}_1(H^1(M, N: C, D)) = 0$ for any C-comodule N.*

(2) *A C-comodule M is (C, D)-injective if and only if $\bar{\zeta}_1(H^1(N, M: C, D)) = 0$ for any C-comodule N.*

Proof. Assume that $\bar{\zeta}_1(H^1(M, N: C, D)) = 0$ for any C-comodule N. Let

$$(**) \qquad 0 \longrightarrow N \longrightarrow K \longrightarrow M \longrightarrow 0$$

be an exact sequence of C-comodules and let $\sigma: M \longrightarrow K$ be a D-comodule map such that $\tau\sigma = 1$. Then by Th. 1.3, $\bar{f}_\sigma \in H^1(M, N: C)$ and $K = (M, N: \bar{f}_\sigma)$ as C-comodule. Since any C-comodule is a D-comodule via Φ, we have

$$K \cong (M, N: \bar{f}_\sigma) \cong (M, N: \bar{\phi}_1(\bar{f}_\sigma))$$

as D-comodule. But, by $K = N \oplus M$ as D-comodule, we have $\bar{\phi}_1(\bar{f}) = 0$, that is,

$$\bar{f}_\sigma \in \mathrm{Ker}\,(\bar{\phi}_1) = \bar{\zeta}_1(H^1(M, N: C, D)) = 0.$$

Thus $\bar{f} = 0$, and the sequence (**) is split as C-comodule.

Conversely, assume that M is (C, D)-projective. For any C-comodule N and for any $\bar{f} \in H^1(M, N: C)$, there exists a C-comodule exact sequence

$$(***) \qquad 0 \longrightarrow N \longrightarrow (M, N : \bar{f}) \longrightarrow M \longrightarrow 0\,.$$

(***) is D-split if and only if $\bar{\phi}_1(f) = 0$. Therefore if $\bar{\phi}_1(\bar{f}) = 0$. then, by the (C, D)-projectivity of M, we have $\bar{f} = 0$. Hence

$$0 = \text{Ker}\ (\bar{\phi}_1) = \bar{c}_1(H^1(M, N : C, D)).$$

This completes the proof of (1). (2) can be proved similarly.

Corollary 2.12. (1) *A C-comodule M is (C, D)-projective if and only if $\bar{c}_1$ is surjective, or equivalently, $\bar{\phi}_1$ is injective for every C-comodule N.*

(2) *A C-comodule N is (C, D)-injective if and only if $\bar{c}_1$ is surjective, or equivalently, $\bar{\phi}_1$ is injective for every C-comodule M.*

Finally, by Cor. 2. 9 and Th. 2. 11, we have the following

Corollary 2.13. (1) *A C-comodule M is (C, D)-projective if and only if $H^1(M, N : C, D) = 0$ for every C-comodule N.*

(2) *A C-comodule N is (C, D)-injective if and only if $H^1(M, N : C, D) = 0$ for every C-comodule M.*

Corollary 2.14. *Let $M = \oplus_\alpha M_\alpha$ as C-comodule. Then M is (C, D)-projective (resp. injective) if and only if each M_α is (C, D)-projective (resp. injective).*

REFERENCES

[1] D. W. JONAH: Cohomology of coalgebras, Mem. Amer. Math. Soc. **52** (1968).
[2] A. NAKAJIMA: Cosemisimple coalgebras and cosepasable coalgebras over coalgebras, Math. J. Okayama Univ. **21** (1979), 125—140.
[3] M. E. SWEEDLER: Hopf algebras, Benjamin, New York, 1969.
[4] D. S. TRUSHIN: Cohomology for modules over associative algebras, J. Alg. **51** (1978), 550—561.

DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY

(Received March 15, 1979)