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A NOTE ON COMMUTATIVE SEPARABLE ALGEBRAS

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In this note, we prove that separability descends by faithful flatness and hence is a local property. We also prove that separability is a punctual property over a semi-local ring.

Throughout this paper, rings and algebras are commutative with identity and ring homomorphisms carry the identity to the identity. In what follows, A denotes a ring with identity 1 and B an A-algebra. B is a separable A-algebra if and only if there exists an element e in $B\otimes_A B$ such that $(b\otimes 1)e=(1\otimes b)e$ for all $b\in B$ and $p(e)=1$, where p is the multiplication map from $B\otimes_A B$ to B ([3, p. 40]). It is easily seen that e is idempotent and unique. This element is called the separability idempotent of B over A, and is invariant under the switch map $B\otimes_A B\rightarrow B\otimes_A B$ given by $b_i\otimes b_j\mapsto b_j\otimes b_i$.

Now, it is well known that if B is separable over A then for any A-algebra C, the C-algebra $B\otimes_A C$ is again separable. Moreover, by [4, Prop. 2. 2 (c)], it is known that if C is a faithfully flat A-algebra and $B\otimes_A C$ is separable over C then B is separable over A, provided that B is finitely generated as an A-algebra. Our main result is that the hypothesis on B is not necessary.

Theorem 1. *Let B be an A-algebra and C a faithfully flat A-algebra. If the C-algebra $B\otimes_A C$ is separable, then B is separable over A.*

Proof. Let ε_0 (resp. ε_1) : $C\rightarrow V=C\otimes_A C$ denote the A-algebra homomorphism defined by $c\mapsto 1\otimes c$ (resp. $c\rightarrow c\otimes 1$). Then, the homomorphisms

$$1\otimes\varepsilon_i:\ U=B\otimes_A C\longrightarrow W=B\otimes_A C\otimes_A C\quad (i=0,\ 1)$$

give rise to the homomorphisms

$$(1\otimes\varepsilon_i)\otimes(1\otimes\varepsilon_i):\ U\otimes_C U\longrightarrow W\otimes_V W\qquad (i=0,\ 1).$$

Moreover, we have the homomorphisms

$$U\xrightarrow{\ \mu\ } U\otimes_{\mathrm{Im}(\varepsilon_i)} V\xrightarrow{\ \nu_i\ }\mathrm{Im}\,(1\otimes\varepsilon_i)\cdot V=W\quad (i=0,\ 1)$$

where $\mu(u)=u\otimes 1$, and $\nu_i(u\otimes v)=(1\otimes\varepsilon_i)(u)v$. Now, in general, if U is separable over T and V is any T-algebra, then the separability idempotent for U over T goes to the separability idempotent for any homomorphic image of $U\otimes_T V$ over V. Applying this to our case, we see that the separability idempotent e' of U over C (which is an element of $U\otimes_C U$) must be sent to 0 under the difference $d=(1\otimes\varepsilon_0)\otimes(1\otimes\varepsilon_0)-(1\otimes\varepsilon_1)\otimes(1\otimes\varepsilon_1)$. Since C is faithfully flat over A, it follows from [2, Lemma 3. 8] that the sequence

$$0\longrightarrow B\otimes_A B\xrightarrow{\rho} B\otimes_A B\otimes_A C\xrightarrow{1\otimes 1\otimes\varepsilon_0\,-\,1\otimes 1\otimes\varepsilon_1} B\otimes_A B\otimes_A C\otimes_A C$$

where $\rho(m)=m\otimes 1$, is exact. From this, one will easily see that the sequence

$$0\longrightarrow B\otimes_A B\xrightarrow{\sigma} U\otimes_C U\xrightarrow{d} W\otimes_V W$$

where $\sigma(b_1\otimes b_2)=(b_1\otimes 1)\otimes(b_2\otimes 1)$, is also exact. Hence there exists an element e in $B\otimes_A B$ so that $\sigma(e)=e'$. Obviously, $p(e)=1$, and $(b\otimes 1)e=(1\otimes b)e$ for all $b\in B$. Thus, B is separable over A, completing the proof.

An application of Th. 1 is the following corollary which shows that separability is a local property.

Corollary 2. *Let B be an A-algebra and let $\{f_1,\cdots,f_n\}$ be a family of elements of A which generates the unit ideal of A. Then B is separable over A if and only if for all i, B_{f_i} is separable over A_{f_i}, where A_{f_i} is the ring of fractions of B having denominators equal to some power of f, and $B_{f_i}=B\otimes_A A_{f_i}$.*

Proof. By the result of [1, Ch. II, Prop. 5. 1. 3], $C=\Pi_{i=1}^n A_{f_i}$ is a faithfully flat A-algebra. From this, the assertion follows immediately.

As a second application of Th. 1, we have the following corollary which shows that separablility is a punctual property provided that the base ring has only a finite number of maximal ideals.

Corollary 3. *Let A be a semi-local ring and B an A-algebra. Then the following conditions are equivalent.*

i) *B is separable over A.*

ii) *$B_\mathfrak{p}$ is separable over $A_\mathfrak{p}$ for each prime ideal $\mathfrak{p}$ of A.*

iii) *$B_\mathfrak{m}$ is separable over $A_\mathfrak{m}$ for each maximal ideal $\mathfrak{m}$ of A.*

Proof. Only iii) ⇒ i) needs proof. Let Ω denote the set of maximal ideals of A. Since Ω is finite, $\Pi_{\mathfrak{m}\in\Omega} A_{\mathfrak{m}}$ is faithfully flat over A by [1, Ch. II, Prop. 3. 3. 10].

Remark 1. By virtue of Corollary 2, we easily see that an A-algebra B is separable if and only if for every prime ideal $\mathfrak{p}$ of A, there exists an element t in $A-\mathfrak{p}$ (the complement of $\mathfrak{p}$ in A) such that B_t is separable over A_t. Moreover, the result of Corollary 3 is a partial generalization of [4, Prop. 2. 5].

Remark 2. By Theorem 1, we see that for A-algebras B, C, if $B\otimes_A C$ is separable over C then B is separable over A, provided that $\{A, C\}$ is one of (1), (2) and (3) :

(1) $A=$ a Noetherian ring, $C=$ the I-adic completion of A where I is an ideal of A contained in the Jacobson radical of A ([1, p. 206]).

(2) $A=$ a local ring, $C=$ the Henselization of A ([3, p. 73]).

(3) $A=$ a coherent ring (e. g., a Noetherian ring), $C=A[[X_1, \cdots, X_n]]$, a formal power series ring over A ([1, p. 49]).

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