COMMUTATIVITY OF CERTAIN RINGS

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Throughout A will represent a ring with a regular element. The set of all regular elements in A and the set of all quasiregular elements in A will be denoted by R and R', respectively. Now, let σ be an automorphism of A. If for each non-zero element y of A there exists an integer $n(y)$ such that $xy = y \cdot x\sigma^{n(y)}$ for all $x \in A$, then A is said to be *σ-commutative*. In case A is σ-commutative, we may assume that each $n(y)$ has been chosen so as to be minimal in its absolute value. Needless to say, a non-zero element y of a σ-commutative ring A is in its center C if and only if $n(y) = 0$. The purpose of this note is to prove the following:

Theorem. *If A is a σ-commutative ring with the prime radical N, then A/N is a commutative reduced ring and A satisfies a polynomial identity* $[[x_1, x_2], x_3] = 0$ *or of the form* $[x_1^k, x_2^k] = 0$.

In advance of proving our theorem, we state a lemma.

Lemma. *Let A be a σ-commutative ring.*

(1) *The classical quotient ring Q of A is also σ-commutative.*

(2) *R and R' generate a commutative (multiplicative) semigroup. In particular, if every zero-divisor of A is nilpotent then A is commutative.*

Proof. (1) Evidently, A has a classical quotient ring Q. As usual, by setting $(xy^{-1})\sigma = x\sigma \cdot (y\sigma)^{-1}$, σ can be extended to an automorphism of Q. It is a routine to check $(xy^{-1})(uv^{-1}) = (uv^{-1})(xy^{-1})\sigma^{n(u)-n(v)}$.

(2) According to (1), we may assume that A coincides with its classical quotient ring. Then, it suffices to show that the unit group $U(A)$ of A is commutative. If $n(y) = 0$ for all $y \in U(A)$ then $U(A)$ is included in the center C. While, if $n(y_0) \neq 0$ for some $y_0 \in U(A)$ then there exists a unit a such that $n(a) = \min\{|n(y)| \mid n(y) \neq 0,\ y \in U(A)\}$. For each $y \in U(A)$, there exist integers q and r such that $n(y) = n(a)q + r$ and $0 \leq r < n(a)$. Now, for any $x \in A$ we have $x(ya^{-q}) = (ya^{-q})x\sigma^r$. By the minimality of $n(a)$, it follows then $r = 0$. Hence, $ya^{-q} \in C$, which means that $U(A)$ is included in the commutative subring $C[a, a^{-1}]$.

Proof of Theorem. First, we shall prove that if P is a proper prime ideal of A, then A/P is a τ-commutative domain with some τ. If $xy \in P$ $(x, y \in A)$ then $xA \cdot yA = xyA \subseteq P$, whence it follows $x \in P$ or $y \in P$. This means that A/P is a domain. We claim here that $P\sigma^{n(y)} = P$ for any $y \in A \backslash P$. In fact, this is evident by $xy = y \cdot x\sigma^{n(y)}$ for all $x \in A$. If $n(y) = 0$ for all $y \in A \backslash P$ then A/P is commutative. While, if $n(y_0) \neq 0$ for some $y_0 \in A \backslash P$ then we can find a minimal positive integer h such that $P\sigma^h = P$. Let τ be the automorphism of A/P induced by σ^h. Now, for each $y \in A \backslash P$ there exist integers q and r such that $n(y) = hq + r$ and $0 \leq r < h$. Since $P = P\sigma^{n(y)} = P\sigma^r$, the minimality of h implies $r = 0$. Hence, $n(y)$ is a multiple of h. This proves that A/P is τ-commutative. Since A/P is a commutative domain by Lemma (2), we readily see that A/N is a commutative reduced ring.

Next, we shall prove the latter assertion. By Lemma (1), we may assume that A coincides with its classical quotient ring. Since $[A, A] \subseteq N \subseteq R'$, if $U(A)$ is included in the center C then $[[A, A], A] = 0$. Henceforth, we assume that $U(A) \not\subseteq C$, and choose a unit a such that $n(a) = \min \{|n(y)| \mid n(y) \neq 0, y \in U(A)\}$. We set $K = C[a, a^{-1}]$ and $k = n(a)$. Now, let y be an arbitrary non-zero element of A. Then there exist integers q and r such that $n(y) = kq + r$ and $0 \leq r < k$. As is easily seen, $x(ya^{-q}) = (ya^{-q}) \cdot x\sigma^r$ for all $x \in A$. If $r = 0$ then $ya^{-q} \in C$, and hence $y^k \in K$. While, if $r > 0$ then $x \cdot (ya^{-q})^k a^{-r} = (ya^{-q})^k a^{-r} \cdot x$ for all $x \in A$. Hence, $(ya^{-q})^k a^{-r} \in C$. Since $U(A) \subseteq K$ (see the proof of Lemma (2)) and $(ya^{-q})^k = y^k u$ with some unit u, it follows eventually $y^k \in K$.

If A is right s-unital then A contains 1, and hence if A is a right $p.p.$ ring in the sense of [3] then A is so in the primary sense [1].

Corollary 1. *If A is a σ-commutative, right $p.p.$ ring then A is commutative.*

Proof. According to Theorem, it suffices to prove that A is a reduced ring. If a is an element of A with $a^2 = 0$ then, by [3, Lemma 5], the right annihilator $r(a)$ of a in A is generated by a central idempotent e. Since a is in $r(a)$, we obtain $a = ae = 0$.

Finally, we can restate [2, Theorem 2.2] as follows:

Corollary 2. *If A is σ-commutative then the following are equivalent:*

1) *A is semiprime, and for any* $x \in A$ *there exists* $x' \in A$ *such that* $r(r(x)) = r(x')$.

2) *For any* $x \in A$ *there exists* $d \in R$ *such that* $dx = x^2$.

3) *The classical quotient ring* Q *of* A *is a regular ring.*

Proof. 1)$\Longrightarrow$2) By Theorem, A is commutative. Obviously, there holds $(x+x')x = x^2$. If $(x+x')y = 0$ then $xy = -x'y \in r(r(x)) \cap r(r(x'))$ $= r(x') \cap r(r(x')) = 0$ by the semiprimeness of A. Hence, $y \in r(x) \cap r(x')$ $= r(x) \cap r(r(x)) = 0$, which means $x + x' \in R$.

2)$\Longrightarrow$3) To be easily seen, A is a reduced ring, and therefore commutative by Theorem. Let xu^{-1} be an arbitrary element of Q, and choose $d \in R$ with $dx = x^2$. We have then $(xu^{-1})(ud^{-1})(xu^{-1}) = x^2d^{-1}u^{-1}$ $= xu^{-1}$

3)$\Longrightarrow$1) By Lemma (1) and Theorem, Q is a commutative regular ring. Given $x \in A$, we have an idempotent $x'u^{-1}$ such that $r_Q(x) = (x'u^{-1})Q$. Hence $r(r(x)) = r_Q(r_Q(x)) \cap A = r_Q(x'u^{-1}) \cap A = r(x')$.

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