THE MONOID STRUCTURE OF GALOIS H-DIMODULE ALGEBRAS INDUCED BY THE SMASH PRODUCT

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Let R be a commutative ring with identity, and let H be a commutative cocommutative finite Hopf algebra over R. In [3], F. W. Long introduced the notion of an H-dimodule algebra as a generalization of that of an abelian group graded algebra which is acted upon by the same group. In the same point of view we shall define a Galois H-dimodule algebra as a generalization of a graded Galois algebra and a Galois H^*-object in the sense of [2]. One of the purposes of this paper is to prove that the set $\mathrm{Gal}_\sharp(R, H)$ of H-dimodule algebra isomorphism classes of Galois H-dimodule algebras has a monoid structure which is induced by the smash product. The group of Galois H^*-objects $\mathrm{Gal}(R, H^*)$ can be naturally regarded as a subgroup of $\mathrm{Gal}_\sharp(R, H)$. In the last section, we shall give two examples for which the monoid $\mathrm{Gal}_\sharp(R, H)$ has group structure.

0. **Preliminaries.** Throughout R will represent a fixed commutative ring with identity 1. We write $\otimes$ and Hom instead of $\otimes_R$ and Hom_R, respectively. Each module is an R-module, each map is R-linear and each algebra is an R-algebra unless otherwise stated. If M is an R-module, M^* denotes $\mathrm{Hom}(M, R)$. We refer to [5] for the theory of Hopf algebras. The comultiplication map and counit map of a Hopf algebra H are denoted by $\Delta: H \to H\otimes H$ and $\varepsilon: H \to R$, respectively. We use the sigma notation $\Delta(h)=\sum_{(h)}h^{(1)}\otimes h^{(2)}$, $h\in H$. The antipode of H is denoted by λ. When the structure map α of X needs explicit mention, we write $\alpha=\alpha_X$.

Let H be a Hopf algebra. An R-algebra A is called an *H-module algebra* if A is an H-module such that the H-action map $\nu: H\otimes A \to A$ is an algebra map. Similarly, A is called an *H-comodule algebra* if A is an H-comodule such that the H-coaction map $\chi: A \to A\otimes H$ is an algebra map. Now let K be another Hopf algebra. An R-algebra A is called an *(H, K)-bimodule algebra* if A is an H-module algebra as well as a K-comodule algebra and

$$(0.1)\qquad \chi\nu=(\nu\otimes 1)(1\otimes\chi): H\otimes A \to A\otimes K.$$

An (H, H)-bimodule algebra is nothing but an *H-dimodule algebra* in the sense of Long [3, Def. 3. 1(iii)].

For an H-module algebra A and an H-comodule algebra B, the *smash product* $A\#B$ is equal to $A\otimes B$ as an R-module but with multiplication

(0. 2) $(a\#b)(c\#d)=\sum_{(b)}a(b^{(1)}c)\#b^{(0)}d,$ where $\chi(b)=\sum_{(b)}b^{(0)}\otimes b^{(1)}.$

Let A be an H-module algebra. An R-subalgebra B of A is called a *sub H-module algebra* of A if B is an H-module algebra with the structure map $\nu_B=\nu_A|H\otimes B$, the restriction of ν_A on $H\otimes B$. A *sub H-comodule algebra* and a *sub (H, K)-bimodule algebra* are defined similarly. Given H-module algebras A and B, a map $f: A\to B$ is called an *H-module algebra map* if f is an algebra map as well as an H-module map. An *H-comodule algebra map* and an *(H, K)-bimodule algebra map* are defined similarly.

The following lemmas are easily seen.

Lemma 0. 1. *Let A, B be (H, K)-bimodule algebras. Then $A\otimes B$ is an $(H\otimes H, K\otimes K)$-bimodule algebra with respect to the following structure:*

(1) *$H\otimes H$-action:* $(h_1\otimes h_2)(a\otimes b)=h_1a\otimes h_2b,$

(2) *$K\otimes K$-coaction:* $\chi(a\otimes b)=\sum_{(a),(b)}a^{(0)}\otimes b^{(0)}\otimes a^{(1)}\otimes b^{(1)}.$

Lemma 0. 2. *Let A, B be H-dimodule algebras. Then $A\#B$ is an $H\otimes H$-dimodule algebra with respect to the following structure:*

(1) *$H\otimes H$-action:* $(h_1\otimes h_2)(a\#b)=h_1a\#h_2b,$

(2) *$H\otimes H$-coaction:* $\chi(a\#b)=\sum_{(a),(b)}a^{(0)}\#b^{(0)}\otimes a^{(1)}\otimes b^{(1)}.$

1. Galois H-dimodule algebras. In this section we show that $\mathrm{Gal}_{\#}(R, H)$ has a monid structure which is induced by the smash product. First we define the notion of a Galois (H, K)-bimodule algebra.

Definition 1. 1. Let H, K be commutative cocommutative finite Hopf algebras. An R-algebra A is called a *Galois (H, K)-bimodule algebra* if

(1) A is an (H, K)-bimodule algebra,

(2) A is a finitely generated projective faithful R-module, and

(3) η_A: $A\otimes A\to \mathrm{Hom}(H, A)$ defined by $\eta_A(a\otimes b)(h)=a(hb)$ is an isomorphism.

If $H=K$, then A is called a *Galois H-dimodule algebra.*

For Galois (H, K)-bimodule algebras A and B, a map $f: A\to B$ is called a *Galois (H, K)-bimodule algebra map* if it is an (H, K)-bimodule algebra map.

In the following, Hopf algebras H, K be always commutative cocommutative finite Hopf algebras.

Remark 1.2. (a) Let $\{h_i, h_i^*\}$ be an R-projective coordinate system of H and let A be an (H, K)-bimodule algebra. Consider the following diagram

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\ \eta_A\ } & \mathrm{Hom}(H, A) \\ & {\scriptstyle\gamma_A}\searrow \quad \swarrow{\scriptstyle\phi} & \\ & A\otimes H^* & \end{array}$$

where $\gamma_A(a\otimes b)=\sum_i a(h_i b)\otimes h_i^*$, $\phi(f)=\sum_i f(h_i)\otimes h_i^*$, $a, b\in A$, $f\in \mathrm{Hom}(H, A)$. Then the above diagram commutes and ϕ is an isomorphism, and so η_A is an isomorphism if and only if so is γ_A. Therefore Def.1.1(3) is equivalent to the following:

(3′) $\gamma_A: A\otimes A\to A\otimes H^*$ defined above is an isomorphism. Note that the map γ_A is independent on the choice of R-projective coordinate systems of H. In the following, $\{h_i, h_i^*\}$ will represent an R-projective coordinate system of H.

(b) If A is a Galois (H, K)-bimodule algebra, then it is easy to see that A is a Galois H^*-object in the sense of [2, Def. 7.3]. Althought the discussion of Galois objects in [2] is limited to commutative algebras, the properties and theories of noncommutative case which we use soon later can be easily proved.

(c) If M is a (left) H-module, then M is a (right) H^*-comodule via $\chi_M(m)=\sum_i h_i m\otimes h_i^*$ $(m\in M)$. Conversely, if M is a (right) H^*-comodule then M is a (left) H-module via $hm=\sum_{(m)} m^{(1)}(h)m^{(0)}$, where $\chi_M(m)=\sum_{(m)} m^{(0)}\otimes m^{(1)}$.

(d) Let G be a finite abelian group. If $H=K=RG$, the group algebra of G over R, then every Galois (H, K)-bimodule algebra is a G-graded algebra which is a G-Galois extension of R.

Proposition 1.3. *Let K be a Hopf algebra which is a free R-module. Let A, B be Galois (H, K)-bimodule algebras. Then the set*

$$A\cdot B=\{\textstyle\sum_i a_i\otimes b_i\in A\otimes B \mid \sum_i ha_i\otimes b_i=\sum_i a_i\otimes hb_i \ \textit{for any}\ h\in H\}$$

is a Galois (H, K)-bimodule algebra with respect to the following structure:

(1) *H-action:* $h(\sum_i a_i\otimes b_i)=\sum_i ha_i\otimes b_i$,

(2) *H-coaction:* $\chi_{A\cdot B}(\sum_i a_i\otimes b_i)=\sum_{i,(a_i),(b_i)} a_i^{(0)}\otimes b_i^{(0)}\otimes a_i^{(1)}b_i^{(1)}$.

Proof. Since A, B are Galois H^*-objects and
$A\cdot B=\{\sum_i a_i\otimes b_i\in A\otimes B\mid \sum_{i,(a_i)}a_i^{(0)}\otimes b_i\otimes a_i^{(1)}=\sum_{i,(b_i)}a_i\otimes b_i^{(0)}\otimes b_i^{(1)}\}$,
$A\cdot B$ is a Galois H^*-object with the H-action given by (1) (see [1, p. 687(6)]). Moreover by (0. 1) and Lemma 0.1, for $\sum_i a_i\otimes b_i$ in $A\cdot B$ we have

$$\sum_{i,j,m} ha_{ij}\otimes b_{im}\otimes k_j\otimes k_m=\sum_{i,j,m}a_{ij}\otimes hb_{im}\otimes k_j\otimes k_m$$

where $\{k_j\}$ is a free basis of K and $\chi_A(a_i)=\sum_j a_{ij}\otimes k_j$, $\chi_B(b_i)=\sum_j b_{ij}\otimes k_j$. Thus $\sum_j a_{ij}\otimes b_{im}$ is in $A\cdot B$ for any j, m and $\chi_{A\otimes B}$ is a map from $A\cdot B$ to $A\cdot B\otimes K\otimes K$. Then an easy computation shows that $A\cdot B$ is a K-comodule algebra with the K-coaction given by (2), and so $A\cdot B$ is a Galois (H, K)-bimodule algebra.

Proposition 1. 4. *Let A, B be Galois H-dimodule algebras. Then $A\# B$ is a Galois $H\otimes H$-dimodule algebra.*

Proof. By Lemma 0. 2, $A\# B$ is an $H\otimes H$-dimodule algebra, and $A\# B$ is clearly a finitely generated projective faithful R-module. Therefore it remains to prove that γ: $(A\# B)\otimes(A\# B)\to(A\# B)\otimes H^*\otimes H^*$ defined by

$$\gamma((a\# b)\otimes(c\# d))=\sum_{i,j}(a\# b\otimes\varepsilon\otimes\varepsilon)(h_ic\# h_jd\otimes h_i^*\otimes h_j^*)$$

is an isomorphism. Since

$$\begin{aligned}(A\# B)\otimes(A\# B)&\cong(A\otimes B)\otimes(A\otimes B)\cong(A\otimes A)\otimes(B\otimes B)\\ &\overset{\gamma_A\otimes\gamma_B}{\cong}(A\otimes H^*)\otimes(B\otimes H^*)\cong(A\otimes B)\otimes H^*\otimes H^*\\ &\cong(A\# B)\otimes H^*\otimes H^*\end{aligned}$$

we can define $\bar\gamma$: $(A\# B)\otimes(A\# B)\to(A\# B)\otimes H^*\otimes H^*$ by

$$\bar\gamma((a\# b)\otimes(c\# d))=\sum_{i,j}a(h_ic)\# b(h_jd)\otimes h_i^*\otimes h_j^* .$$

Then $\bar\gamma$ is an isomorphism and we have

$$\begin{aligned}&\gamma(\textstyle\sum_{(b)}(a\# b^{(0)})\otimes(\lambda(b^{(1)})c\# d)\\ &=\textstyle\sum_{i,j,(b)}a(b^{(1)}h_i\lambda(b^{(2)})c)\# b^{(0)}(h_jd)\otimes h_i^*\otimes h_j^*\\ &=\textstyle\sum_{i,j,(b)}a(h_ib^{(1)}\lambda(b^{(2)})c)\# b^{(0)}(h_jd)\otimes h_i^*\otimes h_j^*\\ &=\textstyle\sum_{i,j,(b)}a(h_i\varepsilon(b^{(1)})c)\# b^{(0)}(h_jd)\otimes h_i^*\otimes h_j^*\\ &=\textstyle\sum_{i,j}a(h_ic)\# b(h_jd)\otimes h_i^*\otimes h_j^*\\ &=\bar\gamma((a\# b)\otimes(c\# d)).\end{aligned}$$

Thus $\mathrm{Im}(\bar{\gamma}) \subseteq \mathrm{Im}(\gamma)$, and so γ is an epimorphism. Counting ranks, γ is seen to be an isomorphism.

Lemma 1.5. *Let A, B be H-dimodule algebras. Regard H as an H-module via μ: $H \otimes H \to H$, the multiplication map of H, and $A \# B$ as an H-comodule via $\chi(a \# b) = \sum_{(a),(b)} a^{(0)} \# b^{(0)} \otimes a^{(1)} b^{(1)}$. Then* $\mathrm{Hom}_{H \otimes H}(H, A \# B)$ *is an H-dimodule algebra with the following structure: For $f, g \in \mathrm{Hom}_{H \otimes H}(H, A \# B)$, $h, x \in H$,*

(1) *H-action:* $(hf)(x) = f(xh)$,

(2) *H-coaction:* $\chi(f) = \sum_i (1 \otimes h_i^*) \chi_{A\#B} f \otimes h_i$,

(3) *algebra structure:* $(f * g)(h) = \sum_{(h)} f(h^{(1)}) g(h^{(2)})$.

Proof. Let h_1, h_2, h, x be in H, and f, g in $\mathrm{Hom}_{H \otimes H}(H, A \# B)$. Then we have

$$(hf)((h_1 \otimes h_2)x) = f((h_1 \otimes h_2)xh) = (h_1 \otimes h_2) f(xh) = (h_1 \otimes h_2)(hf)(x)$$

and

$$\begin{aligned}(h(f * g))(x) &= (f * g)(xh) = \textstyle\sum_{(x),(h)} f(x^{(1)} h^{(1)}) g(x^{(2)} h^{(2)}) \\ &= (\textstyle\sum_{(h)} (h^{(1)} f) * (h^{(2)} g))(x).\end{aligned}$$

Therefore $\mathrm{Hom}_{H \otimes H}(H, A \# B)$ is an H-module algebra. Next we shall show that the H-comodule structure is well defined. Let $f(x) = \sum_j a_j \# b_j$. Then we have

$$\begin{aligned}&(1 \otimes h_i^*) \chi_{A\#B} f((h_1 \otimes h_2)x) \\ &\quad = (1 \otimes h_i^*) \chi_{A\#B} (\textstyle\sum_j h_1 a_j \# h_2 b_j) \\ &\quad = \textstyle\sum_{j,(a_j),(b_j)} h_1 a_j^{(0)} \# h_2 b_j^{(0)} \otimes h_i^*(a_j^{(1)} b_j^{(1)}) \\ &\quad = (h_1 \otimes h_2)(1 \otimes h_i^*) \chi_{A\#B} f(x)\end{aligned}$$

and ρ: $\mathrm{Hom}(H, A \# B \otimes H) \to \mathrm{Hom}(H, A \# B) \otimes H$ defined by $\rho(y) = \sum_i (1 \otimes h_i^*) y \otimes h_i$ is a natural isomorphism. Thus χ: $\mathrm{Hom}_{H \otimes H}(H, A \# B) \to \mathrm{Hom}_{H \otimes H}(H, A \# B) \otimes H$ is well defined. Now we have

$$\begin{aligned}&[\textstyle\sum_{i,n} (1 \otimes h_n^*) \chi_{A\#B}((1 \otimes h_i^*) \chi_{A\#B} f) \otimes h_n \otimes h_i](x) \\ &\quad = \textstyle\sum_{i,n} (1 \otimes h_n^*)(\sum_{j,(a_j),(b_j)} \chi_{A\#B}(a_j^{(0)} \# b_j^{(0)}) \otimes h_i^*(a_j^{(1)} b_j^{(1)}) \otimes h_n \otimes h_i \\ &\quad = \textstyle\sum_{i,j,n,(a_j),(b_j)} a_j^{(0)} \# b_j^{(0)} \otimes h_n^*(a_j^{(1)} b_j^{(1)}) \otimes h_i^*(a_j^{(2)} b_j^{(2)}) \otimes h_n \otimes h_i \\ &\quad = \textstyle\sum_{j,(a_j),(b_j)} a_j^{(0)} \# b_j^{(0)} \otimes a_j^{(1)} b_j^{(1)} \otimes a_j^{(2)} b_j^{(2)} \\ &\quad = (\chi_{A\#B} \otimes 1) \chi_{A\#B} f(x) \\ &\quad = ((1 \otimes \Delta) \chi_{A\#B} f)(x) \\ &\quad = (\textstyle\sum_i (1 \otimes h_i^*) \chi_{A\#B} f \otimes \Delta(h_i))(x)\end{aligned}$$

and

$$
\begin{aligned}
&(\textstyle\sum_i (1 \otimes h_i^*)\chi_{A\#B} f \otimes \varepsilon(h_i))(x) \\
&\quad = \textstyle\sum_{j,(a_j),(b_j)} a_j^{(0)} \# b_j^{(0)} \otimes \varepsilon(a_j^{(1)} b_j^{(1)}) \\
&\quad = \textstyle\sum_j a_j \# b_j = f(x).
\end{aligned}
$$

Therefore χ is an H-comodule structure map. Moreover if $\Delta(x)=\sum_k y_k \otimes z_k$, $f(y_k) = \sum_m c_{km} \# d_{km}$ and $g(z_k) = \sum_p u_{kp} \# v_{kp}$, then

$$
\begin{aligned}
&(\chi(f)\chi(g))(x) \\
&\quad =(\textstyle\sum_i (1 \otimes h_i^*)\chi_{A\#B} f \otimes h_i)(\sum_j (1 \otimes h_j^*)\chi_{A\#B} g \otimes h_j)(x) \\
&\quad = \textstyle\sum_{i,j,k} (1 \otimes h_i^*)\chi_{A\#B} f(x_k)(1 \otimes h_j^*)\chi_{A\#B} g(x_k) \otimes h_i h_j \\
&\quad = \textstyle\sum_{k,m,p,(c_{km}),(d_{km}),(u_{kp}),(v_{kp})} (c_{km}^{(0)} \# d_{km}^{(0)})(u_{kp}^{(0)} \# v_{kp}^{(0)}) \otimes c_{km}^{(1)} d_{km}^{(1)} u_{kp}^{(1)} v_{pk}^{(1)} \\
&\quad = \textstyle\sum_{i,k} (1 \otimes h_i^*)\chi_{A\#B}(f(y_k) g(z_k)) \otimes h_i \\
&\quad = \chi(f * g)(x)
\end{aligned}
$$

and

$$
(\textstyle\sum_i (1 \otimes h_i^*)\chi_{A\#B}\varepsilon \otimes h)(x) = \varepsilon(x) \otimes 1.
$$

Thus χ is an algebra map. Finally we have

$$
\begin{aligned}
&[h(\textstyle\sum_i (1 \otimes h_i^*)\chi_{A\#B} f) \otimes h_i](x) \\
&\quad = \textstyle\sum_i ((1 \otimes h_i^*)\chi_{A\#B} f)(xh) \otimes h_i \\
&\quad = \textstyle\sum_i (1 \otimes h_i^*)\chi_{A\#B}(hf)(x) \otimes h_i \\
&\quad = [\textstyle\sum_i (1 \otimes h_i^*)\chi_{A\#B}(hf) \otimes h_i](x).
\end{aligned}
$$

Hence $\mathrm{Hom}_{H\otimes H}(H, A\# B)$ is an H-dimodule algebra.

Proposition 1.6. *Let H be a Hopf algebra which is a free R-module with a free basis $\{h_i\}$, and let A, B be Galois H-dimodule algebras. Regard* $\mathrm{Hom}_{H\otimes H}(H, A\# B)$ *as an H-dimodule algebra as in Lemma* 1.5. *Then* $\mathrm{Hom}_{H\otimes H}(H, A\# B)$ *is a Galois H-dimodule algebra. Moreover* $\mathrm{Hom}_{H\otimes H}(H, A\# B) \cong (A\cdot B)^{\#}$ *as Galois H-dimodule algebras, where* $(A\cdot B)^{\#}=\{\sum_i a_i \# b_i \in A\# B \mid \sum_i ha_i \# b_i = \sum_i a_i \# hb_i$ *for any* $h \in H\}$ *and the H-action and H-coaction of* $(A\cdot B)^{\#}$ *are those of* $A\cdot B$.

Proof. Let f be in $\mathrm{Hom}_{H\otimes H}(H, A\# B)$, and let $f(1) = \sum_i a_i \# b_i$ be in $A\# B$. Since f is an $H\otimes H$-module map, we have

$$
\textstyle\sum_i ha_i \# b = \sum_i a_i \# hb_i \quad \text{for any } h \in H,
$$

and so $f(1) = \sum_i a_i \otimes b_i$ is in $A\cdot B$. Conversely, let $\sum_i a_i \otimes b_i$ be in

$A \cdot B$. If we define a map $f\colon H \to A\#B$ by $f(h)=\sum_i ha_i \# b_i$, then f is an $H\otimes H$-module map. Therefore the map

$$\rho\colon \mathrm{Hom}_{H\otimes H}(H,\ A\#B) \to A\cdot B$$

defined by $\rho(f)=f(1)$ is an H-module isomorphism. A brief computation shows that ρ is an H-module algebra isomorphism from $\mathrm{Hom}_{H\otimes H}(H, A\#B)$ to $(A\cdot B)^{\#}$, and by Prop. 1. 3 $\mathrm{Hom}_{H\otimes H}(H,\ A\#B)\cong A\cdot B$ is a finitely generated projective faithful R-module. Now we must show that the map

$$\gamma\colon (A\cdot B)^{\#}\otimes (A\cdot B)^{\#} \to (A\cdot B)^{\#}\otimes H^*$$

defined by $\gamma((a\#b)\otimes(c\#d)) = \sum_i (a\#b)(h_ic\#d)\otimes h_i^* = \sum_{i,(b)} a(b^{(1)}h_ic)\# b^{(0)}d\otimes h_i^*$ is an isomorphism, where $\{h_i,\ h_i^*\}$ is a dual basis of H. First we claim that if $a\#b\in(A\cdot B)^{\#}$, then $a\#b_i\in(A\cdot B)^{\#}$, where $\chi_B(b)=\sum_i b_i\otimes h_i$. (Note that $(1\otimes 1\otimes \varepsilon\otimes 1)\chi_{A\#B}(ha\#b) = (1\otimes 1\otimes \varepsilon\otimes 1)\chi_{A\#B}(a\#hb)$.) Since

$$(A\cdot B)^{\#}\otimes (A\cdot B)^{\#}\cong A\cdot B\otimes A\cdot B\cong A\cdot B\otimes H^*\cong (A\cdot B)^{\#}\otimes H^*,$$

we denote the composite of the above maps by $\bar{\gamma}$, that is,

$$\bar{\gamma}((a\#b)\otimes(c\#d))=\sum_i a(h_ic)\# bd\otimes h_i^*\,.$$

Noting that $a\#b_j$ and $\lambda(h_j)c\#d$ are in $(A\cdot B)^{\#}$, we have

$$\gamma(\textstyle\sum_j (a\#b_j)\otimes\lambda(h_j)c\#d)) = \bar{\gamma}((a\#b)\otimes(c\#d))$$

by the same calculation as in the proof of Prop. 1. 4. Therefore $\mathrm{Im}(\bar{\gamma})\subseteq \mathrm{Im}(\gamma)$ and γ is an epimorphism. Counting ranks, γ is seen to be an isomorphism, and thus $(A\cdot B)^{\#}$ is a Galois H^*-object. Since $A\#B$ is an H-comodule algebra via $\chi(a\#b) = \sum_{(a),(b)} a^{(0)}\otimes b^{(0)}\otimes a^{(1)}b^{(1)}\in A\otimes B\otimes H$, $(A\cdot B)^{\#}$ is sub H-dimodule algebra of $A\#B$ by the proof of Prop.1.3. Hence $(A\cdot B)^{\#}$ is a Galois H-dimodule algebra and the map ρ is an H-dimodule algebra map, completing the proof.

The next lemma will be easily seen.

Lemma 1.7. *H^* is a Galois (H,K)-bimodule algebra with respect to the following structure: For $h, x\in H$, $f, g\in H^*$,*

(1) *H-action: $(hf)(x)=f(xh)$,*

(2) *K-coaction is trivial,*

(3) *algebra structure: $(f*g)(h) = \sum_{(h)} f(h^{(1)})g(h^{(2)})$.*

Proposition 1.8. *Let A be a Galois H-dimodule algebra. Then $(A\cdot H^*)^\#$ is isomorphic to A as Galois H-dimodule algebras.*

Proof. We define a map $\phi: (A\cdot H^*)^\# \to A$ by $\phi(\sum_i a_i \# h_i^*) = \sum_i a_i h_i^*(1)$. Then by Lemma 1. 7 (2) and the definitions of the H-action and the H-coaction on $(A\cdot H^*)^\#$, ϕ is seen to be a Galois H-dimodule algebra homomorphism. Thus by [1, Lemma 1. 1], ϕ is an isomorphism.

Theorem 1.9. *Let H be a Hopf algebra which is a free R-module. Let* $\mathrm{Gal}_\#(R, H)$ *be the set of H-dimodule algebra isomorphism classes of Galois H-dimodule algebras. Then* $\mathrm{Gal}_\#(R, H)$ *has the following monoid structure:*

$$[A][B] = [(A\cdot B)^\#] \qquad ([A],\ [B] \in \mathrm{Gal}_\#(R,\ H)),$$

and $[H^*]$ is the identity in $\mathrm{Gal}_\#(R, H)$.

Proof. By Prop. 1. 6, $(A\cdot B)^\#$ is a Galois H-dimodule algebra. Since $A\cdot B = (A\cdot B)^\#$ as R-modules, the associativity of the product is clear by [1, p. 689] and [3, Th. 3. 3]. Moreover by Prop. 1. 8, $[H^*]$ is the identity in $\mathrm{Gal}_\#(R, H)$.

Remark 1.10. Let $\mathrm{Gal}(R, H^*)$ be the set of H-dimodule algebra isomorphism classes of Galois H^*-objects. Let A be a Galois H^*-object. Since A is an H-comodule algebra with the trivial H-coaction, A is an H-dimodule algebra. Therefore we can easily check that the canonical map $f: \mathrm{Gal}(R, H^*) \to \mathrm{Gal}_\#(R, H)$ defined by $f((A)) = [A]$ is a monoid monomorphism.

2. Examples. In this section we give two examples for which $\mathrm{Gal}_\#(R, H)$ has a group structure.

2.1. Let H be a Hopf algebra, and A an H-dimodule algebra. We define $\bar{A}$ to be the R-module A with multiplication given by

$$\bar{a}\cdot\bar{b} = \overline{\textstyle\sum_{(a)}(a^{(1)}b)a^{(0)}}$$

and with H-action and H-coaction inherited from A, where $\bar{a}$ denotes a regarded as an element of $\bar{A}$. Then $\bar{A}$ is really an H-dimodule algebra ([3, Th. 3. 5]).

Lemma 2.1. *Let A be a commutative Galois H-dimodule algebra. Then $\bar{A}$ is a Galois H-dimodule algebra.*

Proof. By the definition of $\bar{A}$, it suffices to prove that $\bar{\gamma}: \bar{A} \otimes \bar{A} \to \bar{A} \otimes H^*$ defined by $\bar{\gamma}(\bar{a} \otimes \bar{b}) = \sum_i \bar{a}\overline{(h_i b)} \otimes h_i^* = \sum_{i,(a)} \overline{(a)^{(1)}(h_i b) a^{(0)}} \otimes h_i^*$ is an isomorphism. First, we assume that H is a free R-module with a free basis $\{h_i\}$. Since A is a Galois H^*-object, there exist elements x_{ij} and y_{ij} in A such that $\sum_i x_{ij}(h_k y_{ij}) = \delta_{j,k}$ (Kronecker's delta). Then

$$\bar{\gamma}(\textstyle\sum_{i,j,(x_{ij})} \overline{x_{ij}^{(0)}} \otimes \overline{\lambda(x_{ij}^{(1)}) y_{ij}} = \sum_{i,j} \overline{(h_k y_{ij}) x_{ij}} \otimes h_k^* = \bar{1} \otimes h_j^*$$

and hence by the definition of $\bar{\gamma}$, $\bar{\gamma}$ is an epimorphism. Counting ranks, $\bar{\gamma}$ is an isomorphism. In case H is general, the localization argument enables to see that $\bar{A}$ is a Galois H-dimodule algebra.

Lemma 2.2 ([4, (7.1) Lemma]). *Let A be R-algebra. If A is projective of rank 2 as an R-module, then A is commutative.*

Lemma 2.3. *Let A be a Galois H-dimodule algebra, and $a \otimes b$ in $A \otimes A$. If $ha \otimes b = a \otimes \lambda(h)b$ for any $h \in H$, then $(ha)b = a(\lambda(h)b)$ is in $A^H = \{c \in A \mid hc = \varepsilon(h)c$ for any $h \in H\}$.*

Proof. Let x be in H. Then

$$\sum_{(x)} x^{(1)} ha \otimes x^{(2)} b = \sum_{(x)} \lambda(x^{(2)}) x^{(1)} ha \otimes b = \varepsilon(x) ha \otimes b.$$

Hence

$$x((ha)b) = \sum_{(x)} (x^{(1)} ha)(x^{(2)} b) = \varepsilon(x)((ha)b) = \varepsilon(x)(a(\lambda(h)b)),$$

namely, $(ha)b = a(\lambda(h)b)$ is in A^H.

Let $G = \langle \sigma \rangle$ be a group of order 2, and $H = RG$. Then RG is a Hopf algebra with the following coalgebra structure and antipode:

$$\Delta(\sigma) = \sigma \otimes \sigma, \qquad \varepsilon(\sigma) = 1, \qquad \lambda(\sigma) = \sigma.$$

If A is an RG-dimodule algebra, then for any $a \in A$, we have

(2.1) $\quad \chi(a) = a_0 \otimes 1 + a_1 \otimes \sigma \qquad (a_0, a_1 \in A).$

Therefore for any $a, b \in A$, we obtain the following

(2.2) $\quad a = a_0 + a_1 \qquad$ (unique),

(2.3) $\quad (ab)_0 = a_0 b_0 + a_1 b_1, \qquad (ab)_1 = a_0 b_1 + a_1 b_0,$

(2.4) $\quad \sigma(a_0) = (\sigma a)_0, \qquad \sigma(a_1) = (\sigma a)_1.$

Let A be a Galois RG-dimodule algebra. By Lemmas 2. 1 and 2. 2, $\bar{A}$ is a Galois RG-dimodule algebra. Then by Lemma 2. 3 and the definition of $(A\cdot\bar{A})^{\#}$, we can define a map ϕ: $(A\cdot\bar{A})^{\#}\to(RG)^{*}$ by $\phi(\sum_i a_i\#\bar{b}_i)(\tau) = \sum_i a_i\tau(b_i)$ $(\tau\in G)$, and we have

$$
\begin{aligned}
&\phi(\textstyle\sum_{i,j}(a_i\#\bar{b}_i)\,(c_j\#\bar{d}_j))\,(\tau)\\
&\quad=\phi(\textstyle\sum_{i,j}(a_ic_j\#\overline{d_j(b_i)_0}+a_i\sigma(c_j)\#\overline{\sigma(d_j)\,(b_i)_1}))\,(\tau)\\
&\quad=\textstyle\sum_{i,j}(a_ic_j\tau(d_j)\tau((b_i)_0)+a_i\sigma(c_j)\tau\sigma(d_j)\tau((b)_1))\\
&\quad=\textstyle\sum_{i,j}(a_i\tau((b_i)_0+\tau(b_i)_1)c_j\tau(d_j))\,(\text{since }\textstyle\sum_ic_j\tau(d_j)\in R\text{ and }G=\langle\sigma\rangle)\\
&\quad=\textstyle\sum_{i,j}a_i\tau_i(b_i)c_j\tau(d_j)\qquad\qquad(\text{by }(2.\,2))\\
&\quad=(\phi(\textstyle\sum_ia_i\#\bar{b}_i))*(\phi(\textstyle\sum_jc_j\#\bar{d}_j))\,(\tau)
\end{aligned}
$$

and

$$
\begin{aligned}
\phi(\sigma(\textstyle\sum_ia_i\#\bar{b}_i))\,(\tau)&=\phi(\textstyle\sum_i\sigma(a_i)\#\bar{b}_i)\,(\tau)=\textstyle\sum_i\sigma(a_i)\tau(b_i)\\
&=\textstyle\sum_ia_i\sigma\tau(b_i)=\textstyle\sum_i\sigma(\phi(a_i\#\bar{b}_i))\,(\tau).
\end{aligned}
$$

Therefore ϕ is an RG-module algebra map. Since $(A\cdot\bar{A})^{\#}$ and $(RG)^{*}$ are Galois $(RG)^{*}$-objects by Prop. 1. 6 and Lemma 1. 7, ϕ is an RG-module algebra isomorphism by [1, Lemma 1. 1]. Next we show that ϕ is an RG-comodule map. Let $a=a_0+a_1$, $b=b_0+b_1$ and $(a\#\bar{b})_0=a_0\#\bar{b}_0+a_1\#\bar{b}_1$. Then $\phi(a\#\bar{b})=\phi((a\#\bar{b})_0)$, because $a\tau(b)=a_0(\tau b)_0+a_1(\tau b)_1+a_0(\tau b)_1+a_1(\tau b)_0$ is in R. Since ϕ is an isomorphism and $\chi_{A\#\bar{A}}((a\#\bar{b}))=(a\#\bar{b})\otimes 1$, $(A\cdot\bar{A})^{\#}$ has the trivial grading. Hence ϕ is an RG-comodule map. Thus we have obtained the following

Theorem 2.4. *Let $G=\langle\sigma\rangle$ be a group of order 2. Then* $\mathrm{Gal}_{\#}(R, RG)$ *is a group.*

Remark 2.5. Let Q be the field of rational numbers, and $a\neq b$ nonzero elements in Q. We set

$$Q[X]/(X^2-a)=Q[x]\quad\text{and}\quad Q[Y]/(Y^2-b)=Q[y],$$

where $x=X+(X^2-a)$ and $y=Y+(Y^2-b)$. Then $Q[x]$ and $Q[y]$ are G-Galois extensions of Q with $\sigma(x)=-x$ and $\sigma(y)=-x$, respectively. The gradings of $Q[x]$ and $Q[y]$ are defined by

$$\chi(x)=x\otimes\sigma\qquad\text{and}\qquad\chi(y)=y\otimes 1.$$

Then it is easy to see that

$$(Q[x]\cdot Q[y])^{\sharp} = Q \oplus Q(x\sharp y),$$
$$(Q[y]\cdot Q[x])^{\sharp} = Q \oplus Q(y\sharp x).$$

If $f\colon Q \oplus Q(x\sharp y) \to Q \oplus Q(y\sharp x)$ is a Q-algebra isomorphism, then

$$f(x\sharp y) = r + s(y\sharp x) \qquad (r, s \in Q \text{ and } s \neq 0)$$

and hence

$$ab = r^2 + 2rs(y\sharp x) - s^2\, ab.$$

Since $2s \neq 0$, we have $r = 0$ and $s^2 = -1$, which is a contradiction. Thus $A \not\cong B$, which means that $\mathrm{Gal}_{\sharp}(Q, QG)$ is not abelian. While, it is known that $\mathrm{Gal}(Q, QG)$ is abelian.

2.2. Let k be a field of characteristic 2, and $H = k \oplus k\delta$ a free k-module with a free basis $\{1, \delta\}$. Then H is a Hopf algebra with the following structure:

$$\delta^2 = 0, \qquad \Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta, \qquad \varepsilon(\delta) = 0, \qquad \lambda(\delta) = \delta.$$

If A is an H-dimodule algebra, then for any $a, b \in A$, we have

(2.5) $\quad \delta(ab) = (\delta a)b + a(\delta b),$

(2.6) $\quad \chi(a) = a \otimes 1 + a_1 \otimes \delta, \qquad \delta(a_1) = a_1 \otimes 1 \ (a_1 \in A),$

(2.7) $\quad (ab)_1 = a_1 b + a b_1,$

(2.8) $\quad \chi(\delta a) = \delta a \otimes 1 + \delta a_1 \otimes \delta, \quad \delta(a)_1 = \delta a_1.$

(2.9) $\quad a_1 = 0$ if $a \in k$.

Let A be a Galois H-dimodule algebra. By Lemmas 2.1 and 2.2, $\bar{A}$ is a Galois H-dimodule algebra. If $\sum_i a_i \sharp \bar{b}_i$ is in $(A\cdot \bar{A})^{\sharp}$, then it is easy to see that

(2.10) $\quad \sum_i a_i b_i,\ \sum_i (\delta a_i) b_i = \sum_i a_i(\delta b_i) \in k = A^H = \{a \in \bar{A} \mid \delta a = 0\},$

(2.11) $\quad 0 = (\sum_i a(\delta b_i))_1 = \sum_i (a_i)_1(\delta b_i) + \sum_i a_i(\delta b_i)_1,$

(2.12) $\quad 0 = \sum_i \delta((\delta a_i) b_i) = \sum_i (\delta a_i)(\delta b_i).$

Define a map $\phi\colon (A\cdot \bar{A})^{\sharp} \to H^*$ by $\phi(\sum_i a_i \sharp \bar{b}_i)(h) = \sum_i a_i h(b_i)\ (h \in H)$.

For $\sum_j c_j \sharp \bar{d}_j \in (A\cdot \bar{A})^{\sharp}$, we have

$$(\textstyle\sum_i a_i \sharp \bar{b}_i)(\sum_j c_j \sharp \bar{d}_j)$$
$$= \textstyle\sum_{i,j}(a_i c_j \sharp \overline{d_j b_i} + a_i c_j \sharp \overline{(\delta d_j)(b_i)_1} + a_i(\delta c_j) \sharp \overline{d_j (b_i)_1})$$
$$= \textstyle\sum_{i,j}(a_i c_j \sharp \overline{d_j b_i} + a_i \delta(c_j d_j) \sharp \overline{(b_i)_1}) \text{ (since } \delta c_j,\ \delta d_j \in k \text{ and (2.5))}$$

$$= \sum_{i,j} a_i c_j \# \overline{d_j b_i} \qquad \text{(by (2. 10))}.$$

Hence we obtain

$$\begin{aligned} \phi((\textstyle\sum_i a_i \# \bar{b}_i)(\sum_j c_j \# \bar{d}_j))(1) &= \textstyle\sum_{i,j} a_i (c_j d_j) b_i \\ &= \textstyle\sum_{i,j} a_i b_i c_j d_j \qquad \text{(by (2. 10))} \\ &= \phi(\textstyle\sum_i a_i \# \bar{b}_i) * \phi(\sum_j c_j \# \bar{d}_j)(1) \end{aligned}$$

and

$$\begin{aligned} \phi((\textstyle\sum_i a_i \# \bar{b}_i)(\sum_j c_j \# \bar{d}))(\delta) &= \textstyle\sum_{i,j} a_i c_j \delta(d_j b_i) \\ &= \textstyle\sum_{i,j}(a_i(\delta b_i) c_j d_j + a_i b_i c_j (\delta d_j)) \qquad \text{(by (2. 10)} \\ &= \phi(\textstyle\sum_i a_i \# \bar{b}_i) * \phi(\sum_j c_j \# \bar{b}_j)(\delta). \end{aligned}$$

Moreover

$$\begin{aligned} \phi(\delta(\textstyle\sum_i a_i \# \bar{b}_i))(1) &= \textstyle\sum_i (\delta a_i) b_i \\ &= \textstyle\sum_i a_i (\delta b_i) \qquad \text{(by (2. 10))} \\ &= [\delta(\phi(\textstyle\sum_i a_i \# \bar{b}_i))](1) \end{aligned}$$

and

$$\begin{aligned} \phi(\delta(\textstyle\sum_i a_i \# \bar{b}_i))(\delta) &= \textstyle\sum_i (\delta a_i)(\delta b_i) = 0 \qquad \text{(by (2. 12))} \\ &= [\delta(\phi(\textstyle\sum_i a_i \# \bar{b}_i))](\delta). \end{aligned}$$

Therefore ϕ is an H-module algebra homomorphism, and by [1, Lemma 1. 1] ϕ is an isomorphism. Finally we show that ϕ is an H-comodule homomorphism.

$$\chi\phi(\textstyle\sum_i a_i \# \bar{b}_i)(h \otimes 1) = \sum_i a_i h(b_i) \otimes 1,$$

$$\begin{aligned} (\phi \otimes 1)\chi(\textstyle\sum_i a_i \# \bar{b}_i)(h \otimes 1) &= \textstyle\sum_i a_i h(b_i) \otimes 1 + (\sum_i (a_i)_1 h(b_i) + a_i (h b_i)_1) \otimes \delta \\ &= \textstyle\sum_i a_i h(b_i) \otimes 1 \qquad \text{(by (2. 9) and (2. 11))}. \end{aligned}$$

Thus we have proved the following

Theorem 2. 6. *Let H be the Hopf algebra defined above. Then $\mathrm{Gal}_t(k, H)$ is a group.*

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