ON POWERS OF ARTINIAN RINGS WITHOUT IDENTITY

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0. **Introduction.** Throughout this paper an Artinian (Noetherian) ring means a left Artinian (Noetherian) ring, i. e., an associative ring with minimum (maximum) condition on left ideals. The existence of an identity is not assumed.

Recently L. S. Levy [3] proved that there is a surprising abundance of indecomposable Artinian, non-Noetherian rings; moreover they can be nonnilpotent. Here an indecomposable ring means a ring which is not the ring-direct sum of two nonzero rings, and this restriction aims to rule out uninteresting trivial cases. According to Hopkins' famous theorem ([2], p. 728]), every Artinian, non-Noetherian ring can not have a left or right identity, because an Artinian ring A is necessarily Noetherian if A contains such an identity. Therefore we have a large class of nonnilpotent rings which can not contain a left or right identity by the nature of themselves. The present paper is motivated by this interesting result.

Let A be any Artinian ring, and consider the descending chain of left ideals: $A \supseteq A^2 \supseteq A^3 \supseteq \cdots$. Then after a finite number of terms we have equalities only. We are interested in the subrings A^k. We shall prove that all A^k for $k \geq 2$ are Artinian and Noetherian even if A is non-Noetherian. It will be further proved that if $A \neq A^2$ then every A^k can not contain a (two-sided) identity.

As is well known, in an Artinian, non-Noetherian ring A the additive group of A contains a divisible torsion subgroup ([1], p. 285). There exists a unique maximal divisilbe, torsion subgroup D of A. The subgroup D is contained in the total annihilator W of A, i. e., $DA=0=AD$ ([1], p. 281). We shall consider such a ring A and prove the following theorems. But, N denotes the radical of A.

First, if A is indecomposable, then $A \neq A^2$, $N^2 \neq 0$, $A^2 \cap D \neq 0$ and A^2 contains no left or right identity. Next, according to Levy [3], if A is indecomposable then the ring $S = A/D$ can not have a left or right identity. However, it can be proved more generally that A/W can not have a left or right identity, whether A may be non-Noetherian or not, provided that A is indecomposable. Further, if $S = S^2$ and if A is indecomposable, then every A^k can not contain a left or right identity.

1. Every A^k is Artinian.

Theorem 1. *If A is an Artinian (Noetherian) ring, then every subring A^k is Artinian (Noetherian).*

Proof. We state the proof only for Artinian case. The slight modification needful in Notherian case is obvious.

Assume that A^k is Artinian for some integer k. Let $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ be any descending chain of left ideals of A^{k+1}. We claim that after a finite number of terms only the equality holds.

Consider first the descending chain of left ideals of A^k, $A^kL_1 \supseteq A^kL_2 \supseteq \cdots$. Then by assumption there exists an integer m such that $A^kL_m = A^kL_{m+1} = \cdots = M$.

Consider next the following two chains:

$$L_m + M \supseteq L_{m+1} + M \supseteq \cdots,$$
$$L_m \cap M \supseteq L_{m+1} \cap M \supseteq \cdots.$$

All $L_j + M$ and $L_j \cap M (j = m, m+1, \cdots)$ are left ideals of A^k, because

$$A^k (L_j + M) = A^k(L_j + A^kL_j) \subseteq M + A^{k+1}L_j \subseteq M + L_j,$$

and similarly $A^k (L_j \cap M) \subseteq M \cap L_j$. Therefore there exists an integer $n \geq m$ such that

$$L_n + M = L_{n+1} + M = \cdots,$$
$$L_n \cap M = L_{n+1} \cap M = \cdots.$$

Then, using the modular law, we can have

$$L_n = L_n \cap (L_n + M) = L_n \cap (L_{n+1} + M)$$
$$= L_{n+1} + (L_n \cap M) = L_{n+1} + (L_{n+1} \cap M) = L_{n+1}.$$

The proof can be now completed by induction on k.

2. Principal Peirce decompositions of A and A^k. Let A be any nonnilpotent Artinian ring, and N the radical of A. Then the identity of the ring A/N can be lifted to an idempotent e of A, which will be called a principal idempotent of A. We have the Peirce decompositions:

$$A = Ae + L, \quad A = eA + R,$$

where L is the left annihilator of e in A, and R the right annihilator of e in A. Naturally both L and R are contained in N, and so we have

$$N = Ne + L, \quad N = eN + R.$$

Since $R = Re + R \cap L$ and $L = eL + L \cap R$,

$$(1) \qquad \begin{aligned} A &= eAe + Re + eL + L \cap R, \\ N &= eNe + Re + eL + L \cap R. \end{aligned}$$

We call (1) the principal Peirce decomposition of A with respect to e. Let $T = L \cap R$. Then we have

$$RL = (Re + T)(eL + T) = ReL + T^2,$$
$$RT^k = (Re + T)T^k = T^{k+1} = T^kL$$

for all positive integers k.

Theorem 2. *Let A be any nonnilpotent Artinian ring, and let $T_k = ReL + T^k$ $(k \geq 1)$. Then there hold the following:*

(i) $A^k = eAe \oplus Re \oplus eL \oplus T_k$.

(ii) *Let* $N_k = eNe \oplus Re \oplus eL \oplus T_k$. *Then N_k is the radical of A^k.*

(iii) $N_k = A^{k-1}N + NA^{k-1}$. *But, here* $k \geq 2$.

(iv) $A^k/N_k \cong A/N$ *(a ring-isomorphism)*.

Remark. Here the notation $\oplus$ means a module-direct sum, while $+$ means merely a linear sum. However, when it is self-evident and there is no fear of confusion, we write $+$ also for $\oplus$.

Proof. (i) Recall (1). Then it is obvious that (i) holds for $k = 1$. Therefore it can be proved by induction on k. Assume that it holds for some integer k. Then we have

$$\begin{aligned} A^{k+1} &= (eA + R)(eAe + Re + eL + ReL + T^k) \\ &= eA(eAe + Re) + R(eAe + Re) \\ &\quad + eA(eL + ReL + T^k) + R(eL + ReL + T^k) \\ &= eAe + Re + eL + ReL + T^{k+1}. \end{aligned}$$

(ii) – (iv) We have

$$\begin{aligned} A^{k-1}N + NA^{k-1} &= (eAe + Re + eL + ReL + T^{k-1})(Ne + L) \\ &\quad + (eN + R)(eAe + Re + eL + ReL + T^{k-1}) \\ &= (eAe + eL)Ne + (Re + ReL + T^{k-1})Ne \\ &\quad + (eAe + eL)L + (Re + ReL + T^{k-1})L \\ &\quad + eN(eAe + Re) + eN(eL + ReL + T^{k-1}) \\ &\quad + R(eAe + Re) + R(eL + ReL + T^{k-1}). \end{aligned}$$

Deleting redundant terms, we get

$$A^{k-1}N + NA^{k-1} = eNe + eL + Re + ReL + T^k = N_k.$$

Clearly N_k is a nilpotent two-sided ideal of A^k. Moreover it is easy to see the following ring-isomorphisms:

$$A^k/N_k \cong eAe/eNe \cong A/N.$$

Hence N_k is the radical of the Artinian ring A^k.

Remember that (i) is the principal Peirce decomposition of A^k with respect to e.

Consider the descending chain of left ideals: $A \supseteq A^2 \supseteq A^3 \supseteqq \cdots$. Let ρ be the nilpotency exponent of N. Then $T^\rho = 0$, because $T = L \cap R \subseteq N$, and so

$$(2) \qquad A^\rho = eAe + Re + eL + ReL.$$

Besides, $A^k = A^\rho$ for all $k > \rho$.

Theorem 3. *For a nonnilpotent Artinian ring A, we have $A^k = A^{k+1}$ if and only if $T^k \subseteq ReL$.*

Proof. Clearly $A^k = A^{k+1}$ is equivalent to $A^k = A^\rho$, which holds if and only if $ReL + T^k = ReL$. This is equivalent to $T^k \subseteq ReL$.

Theorem 4. *Let A be a nonnilpotent Artinian ring. Then there hold the following:*

(i) $A = A^2$ *if and only if* $R \cap L = RL$.

(ii) $A^2 = A^3$ *if and only if* $RL = ReL$.

Proof. (i) By Theorem 3, $A = A^2$ if and only if $R \cap L \subseteq ReL$. Note the following relation:

$$R \cap L \supseteq RL \supseteq ReL.$$

Then (i) is obvious.

(ii) By Theorem 3, $A^2 = A^3$ if and only if $T^2 \subseteq ReL$, which is equivalent to $RL = ReL$, because $RL = ReL + T^2$.

3. Every A^k ($k \geq 2$) is Noetherian. In the previous paper [4], we have proved the following theorem: An Artinian ring A is Noetherian if and only if R is a finite set. But, let $R = A$ if A is nilpotent.

Now, recall the theorem of Hopkins that Re is a finite set, which was also reproved in [4]. Then, since $R = Re + R \cap L = Re + T$, we can restate the above theorem as follows.

Theorem 5. *An Artinian ring A is Noetherian if and only if T is* *a finite set. But, let* $T = A$ *if* A *is nilpotent.*

For further study we cite also the following result of the previous paper [4]: In an Artinian ring A, the group $(R, +)$ satisfies the minimum condition on subgroups. But, let $R = A$ if A is nilpotent.

Theorem 6 (cf. the proof of [3], Proposition 2. 6). *In a nonnilpotent Artinian ring* A, ReL *is a finite set.*

Proof. The elements of ReL are finite sums $\sum a_i b_i$, $a_i \in Re$, $b_i \in eL$. Therefore ReL forms a subgroup of the group $(R, +)$, and it satisfies the minimum condition on subgroups. Besides, the group ReL is of bounded order, because Re is a finite subgroup of $(R, +)$. Now, as is well known, an additive Abelian group G of bounded order is a direct sum of cyclic groups. If G moreover satisfies the minimum condition, then the number of the summands must be finite, and hence G is finite. By this reason, ReL is finite.

Theorem 7 (Levy [3], p. 281). *Let* A *be an Artinian ring. If* $A = A^2$ *then* A *is Noetherian.*

Proof. Clearly we can assume that A is nonnilpotent. If $A = A^2$, then we have $T = RL = ReL$ by Theorem 4. Hence T is finite, and so A is Noetherian by Theorem 5.

According to Fuchs [1], an Artinian ring A is Noetherian if and only if the group $(A, +)$ contains no quasicyclic p-group. A quasicyclic p-group is a group of type $Z(p^\infty)$, and so it is a divisible torsion group. Furthermore such a group belongs to the total annihilator W of A.

Let A be an Artinian, non-Noetherian ring. Then by the above theorem, A contains a divisible torsion subgroup, and the subgroup is contained in R, because naturally $W \subseteq R$.

Recall now the following theorem of Kuroš ([1], p. 65): The subgroups of an additive Abelian group G satisfy the minimum condition if and only if G is a direct sum of a finite number of quasicyclic and/or cyclic p-groups.

Since the group $(R, +)$ satisfies the minimum condition on subgroups, the theorem of Kuroš can be applied to the subgroup T of R. Thus we have

$$(3) \qquad T = T_0 + D, \qquad T_0 \cap D = 0,$$

where T_0 is a finite subgroup and D is the direct sum of a finite number of quasicyclic p-groups. Therefore we can write

$$A = A_0 + D, \qquad A_0 \cap D = 0,$$

where $A_0 = eAe + Re + eL + T_0$. Then it is obvious that D is a unique maximal divisible, torsion subgroup of the additive group of A.

Remark. The following theorem of Szász-Levy ([3], p. 281) is worthy of note: If an Artinian, non-Noetherian ring A is indecomposable, then the additive group of A is primary. It implies that the subgroups of type $Z(p^\infty)$ of A are of the same prime p.

Theorem 8. *Let A be any Artinian ring. Then every subring A^k is Notherian for $k \geq 2$.*

Proof. If A itself is Noetherian, then every A^k is Noetherian. It is already proved in Theorem 1. Therefore there remains the case where A is non-Noetherian. In this case the group $(A, +)$ contains the maximal divisible, torsion subgroup D.

First, assume that A is nonnilpotent, and consider the Peirce decompositions (1) of A and (i) of A^k in Theorem 2. The term T can be written as (3). Note that T_0 is a finite subgroup of T. Then obviously $T^k = T_0^k$, and it is finite. Therefore, by Theorem 6, $T_k = ReL + T^k$ is finite, too. Henc A^k is Noetherian by Theorem 5.

Next, assume that A is nilpotent. Then, since the group $(A, +)$ satisfies the minimum condition, by the theorem of Kuroš we can write $A = A_0 + D$, $A_0 \cap D = 0$, where A_0 is a finite subgroup. Then clearly $A^k = A_0^k$, and it is finite. Hence A^k is Noetherian.

4. Existence of an identity.

Theorem 9. *Let A be an Artinian ring such that $A \neq A^2$. Then every A^k can not contain an identity.*

Proof. In case A is nilpotent, it is trivial. Further, if A is decomposable and $A \neq A^2$, then for some indecomposable direct summand A_i we have $A_i \neq A_i^2$. Therefore it is clear that A may be assumed to be indecomposable and nonnilpotent.

Under this assumption, suppose that A^k contains an identity e'. We first claim that e' is a principal idempotent of A.

Consider (1) and write an element a of A as

$$a = a_{11} + a_{01} + a_{10} + a_{00},$$

$$a_{11} \in eAe, \ a_{01} \in Re, \ a_{10} \in eL, \ a_{00} \in T.$$

Then $a + N = a_{11} + N$. Since $a_{11} \in A^k$, we have

$$(e' + N)(a + N) = (e' + N)(a_{11} + N)$$
$$= e'a_{11} + N = a_{11} + N = a + N.$$

Therefore $e' + N$ is the identity of the semisimple ring A/N, and hence e' is a principal idempotent of A.

Take now e' as the principal idempotent e for (1), and consider (i) of Theorem 2. Then we must have $A^k = eAe$, and so $Re = 0$ and $eL = 0$. Consequently,

$$A = eAe + T, \quad A^2 = eAe + T^2.$$

Here $T \neq 0$, because $A \neq A^2$. Moreover, $A = eAe + T$ is clearly a ring-direct sum. It is a contradiction.

Theorem 10. *Let A be a nonnilpotent Artinian, non-Noetherian ring. If A is indecomposable, then there hold the following:*

(i) $A \neq A^2$.

(ii) $N^2 \neq 0$.

(iii) $A^2 \cap D \neq 0$.

(iv) *A^2 contains no left or right identity.*

Proof. (i) It is clear by Theorem 7.

(ii) Suppose $N^2 = 0$. Then by (i) of Theorem 2, we have $A^2 = eAe + Re + eL$, because both ReL and T^2 are contained in N^2. Therefore we have $A = A^2 + T$ and $A^2 T = 0 = TA^2$. It follows that A is the ring-direct sum of A^2 and T, contradictory to assumption.

(iii) Suppose $A^2 \cap D = 0$. Note that D is a direct summand of A and that the complementary summand can be so chosen as to contain A^2 ([1], p. 63). Therefore we can write

$$A = B + D, \quad B \cap D = 0, \quad A^2 \subseteq B.$$

Then $B^2 = (B + D)^2 = A^2 \subseteq B$. Hence $A = B \oplus D$, a ring-direct sum. It is a contradiction.

(iv) By reason of (iii) A^2 contains a nonzero element of D. It annihilates every element of A^2. Therefore A^2 can not contain a left or right identity.

Theorem 11. *Let A be a nonnilpotent Artinian ring with radical N of nilpotency exponent $\rho \geq 2$. If $A^{\rho-1} \neq A^\rho$ and $A^{\rho-1}$ is indecomposable, then A^ρ contains no left or right identity.*

Proof. Suppose that A^ρ contains a left identity e'. By the same argument as that in the proof of Theorem 9, we can see that e' is a principal idempotent of A. Take e' as e for (1), and consider (2). Then we must have $A^{\rho} = eAe + eL$, and

$$A^{\rho-1} = (eAe + eL) + T^{\rho-1} = A^{\rho} + T^{\rho-1}.$$

Here $(eAe)T^{\rho-1} = 0 = T^{\rho-1}(eAe)$, $(eL)T^{\rho-1} \subseteq N^{\rho} = 0$ and $T^{\rho-1}(eL) = 0$. Therefore $A^{\rho-1}$ is the ring-direct sum of $A^{\rho} \neq 0$ and $T^{\rho-1} \neq 0$. It is a contradiction.

5. The ring $S = A/D$.

Theorem 12. *Let A be a nonnilpotent Artinian ring with the total annihilator $W \neq 0$. If A is indecomposable, then the ring A/W contains no left or right identity.*

Proof. Suppose that A/W contains a left identity $f + W$, $f \in A$. Then $f^2 \equiv f$ (mod W), and f acts on the elements of A as a left identity modulo W. Since W is contained in the radical N of A, the element f acts on A as a left identity moduo N. Hence $f + N$ is the identity of A/N.

Let $w = f^2 - f$ and $e = f + w$. Then $e^2 = f^2 = f + w = e$, and $e \equiv f$ (mod N). Therefore e is a principal idempotent of A. Consider the Peirce decomposition of A with respect to e, and let it be (1). Then we have naturally $W \subseteq R \cap L$.

For every element x of R,

$$fx = (e - w)x = ex - wx = 0.$$

On the other hand we have $fx \equiv x$ (mod W). Therefore $x \in W$, and we see $R \subseteq W$. It follows that $Re = 0$ and $R \cap L \subseteq W$. Hence $W = R \cap L$. Therefore we have

$$A = (eAe + eL) \oplus W.$$

This is a ring-direct sum. It is a contradiction.

Theorem 13 (Levy [3], p. 290). *If a nonnilpotent Artinian, non-Noetherian ring A is indecomposable, then the ring $S = A/D$ contains no left or right identity.*

Proof. Suppose that S has a left identity $f + D$. Then f acts on the elements of A as a left identity modulo D, and also modulo W, because $D \subseteq W$. Then $f + W$ is a left identity of A/W, contradictory to Theorem 12.

Theorem 14. *Let A be a nonnilpotent Artinian, non-Noetherian ring, and let $S = A/D$. Then $S = S^2$ if and only if $T = ReL + D$.*

Proof. Consider (1) and (2):

$$A = eAe + Re + eL + T,$$
$$A^\rho = eAe + Re + eL + ReL.$$

The condition $S = S^2$ is equivalent to $S = S^\rho$, i. e.,

$$A/D = (A/D)^\rho = (A^\rho + D)/D.$$

Therefore, if $S = S^2$, then we have $A = A^\rho + D$, which implies $T = ReL + D$.

Theorem 15. *Let $S = A/D$ under the same assumption as that in Theorem* 14. *If $S = S^2$, and if A is indecomposable, then every A^k can not contain a left or right identity.*

Proof. By Theorem 14, $T = ReL + D$. Here we have $ReL \cap D \neq 0$, because otherwise we would have

$$A = A^\rho \oplus D \qquad \text{(a ring-direct sum)},$$

contradictory to assumption. Therefore every A^k contains a nonzero element of D, and hence it can not contain a left or right identity.

As for the case of $S \neq S^2$, it is not yet certain to the authors whether $A^\rho (= A^{\rho+1} = \cdots)$ can contain a one-sided identity or not. Remember that A^ρ is an idempotent Artinian, Noetherian ring. In general, some of such rings can contain a one-sided identity, others can not.

Example. Let $K = Z/pZ$ (p a prime), and consider the ring A expressed as follows:

$$A = \begin{pmatrix} K & 0 & 0 \\ K & 0 & 0 \\ K & K & K \end{pmatrix}.$$

It is an Artinian, Noetherian ring with $A = A^2$. The radical of A is

$$N = \begin{pmatrix} 0 & 0 & 0 \\ K & 0 & 0 \\ K & K & 0 \end{pmatrix}.$$

We have $AN \neq N$ and $NA \neq N$. Therefore A contains neither left nor right identity.

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