ON GENERALIZATIONS OF V-RINGS AND REGULAR RINGS

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Introduction. Rings whose proper cyclic right modules are injective (called *right PCI-rings*) are studied in [1] and [7]. In this paper we consider rings whose cyclic left modules are either p-injective or projective (called *left CPP-rings*) and rings whose cyclic left modules are either p-injective or flat (called left *CPF-rings*). Left *PCI*-rings and regular rings are obviously left *CPP*-rings but the converse is not true in either case. The following results are proved in Theorem 2: (1) If A is a left *CPP*-ring then A is a left *CPF*-ring that $l(b)$ is principal for each $b \in A$; (2) If A is a *CPF*-ring such that $l(b)$ is principal for each $b \in A$ then A is a fully left idempotent, left $p.p.$ ring. We then continue our study of regular rings and (generalizations of) V-rings in [12] – [15]. (For rings without identity, cf. [3], [4], [10] and [11].) The characterization of a regular ring as an absolutely flat ring [8] may be weakened as follows: A is regular iff every cyclic singular left A-module is flat (Theorem 5). This will provide a generalization of [14, Theorem 9 (ii)] (cf. also [11, Theorem 2 6)]). A well-known theorem of Villamayor [9] states that A is a left V-ring iff every left ideal of A is an intersection of maximal left ideals of A. We here prove an analogous result for rings whose simple left modules are f-injective (Theorem 6). Two further characteristic properties of left V-rings are also given (Theorem 8).

Throughout A will represent an associative ring with identity, and A-modules are unitary. A left A-module M is called *p-injective* (resp. *f-injective*) if for any principal (resp. finitely generated) left ideal I of A and any $g: {}_AI \to {}_AM$ there exists $y \in M$ such that $g(b) = by$ for all $b \in I$. Recall that the left singular submodule of ${}_AM$ is $Z(M) = \{z \in M \mid l(z) \text{ is essential in } {}_AA\}$. M is called *singular* (resp. *non-singular*) if $Z(M) = M$ (resp. $Z(M) = 0$). A is said to be *fully left idempotent* if every left ideal of A is idempotent. As usual, A is called a *left $p.p.$ ring* if every principal left ideal of A is projective. Finally, given a left ideal I of A, the intersection of all maximal left ideals of A containing I is denoted by I^*.

1. *CPF*-rings and regular rings.

Left *PCI*-rings are obviously left *CPF*-rings but the converse is not true (cf. [7, Theorem 14]). Also, A

is not necessarily regular when A is a left *CPF*-ring such that $l(a)$ is principal for any $a \in A$. However, we shall prove that such rings are fully left idempotent, left $p.p.$ rings.

Lemma 1. *Let $I = Aa$ be a principal left ideal of A.*

(1) *If I is p-injective then I is a direct summand of ${}_AA$.*

(2) *If A/I is flat then A/I is projective and I is a direct summand of ${}_AA$.*

(3) *If I is projective and A/I is p-injective then $I = I^2$.*

Proof. (1) is easy (cf. [12, Lemma 2]), and (2) is a direct consequence of [2, Corollary to Proposition 2. 2].

(3) Define $f: {}_AA/I \to {}_AI/I^2$ by $x + I \to xa + I^2$. Since I is projective, there exists $g: {}_AI \to_A A/I$ such that $fg(y) = y + I^2$. Also, there exists an element b in A such that $g(y) = yb + I$. Then $y + I^2 = fg(y) = f(yb + I) = yba + I^2 = I^2$ for any $y \in I$, namely, $I = I^2$.

Theorem 2. *Consider the following statements:*

1) *A is a left CPP-ring.*

2) *A is a left CPF-ring and $l(b)$ is principal for any $b \in A$.*

3) *A/Ab is either p-injective or flat, and $l(b)$ is principal for any $b \in A$.*

4) *A is a fully left idempotent, left p. p. ring.*

Then i) implies $i + 1$) for $i = 1, 2, 3$.

Proof. By Lemma 1 (1), 1) $\Longrightarrow$ 2) $\Longrightarrow$ 3). Now, assume 3), and let $l(b) = Ac$. If $Ab(\simeq A/Ac)$ is p-injective or A/Ab is flat then Ab is a direct summand of ${}_AA$ (Lemma 1 (1) and (2)). If $Ab\ (\simeq A/Ac)$ is flat and A/Ab is p-injective, then Ab is projective and idempotent (Lemma 1 (2) and (3)), proving 4).

If A is a $P.I.$-ring or a left semi-Artinian ring, then A is regular iff A is fully left idempotent [8, Theorems 16 and 17]. This together with Theorem 2 yields the following

Corollary 3. *If A is a P. I.-ring or a left semi-Artinian ring then the following are equivalent:*

1) *A is a regular ring.*

2) *A is a left CPP-ring.*

3) *A is a left CPF-ring such that $l(b)$ is principal for any $b \in A$.*

Rings whose singular left modules are p-injective need not be regular (cf. [5]). However, the next proposition shows that such rings are left $p.p.$ rings.

Proposition 4. *The following conditions are equivalent*:

1) *A is a left $p.p.$ ring.*

2) *For every left A-module M, $\hat{M}/M$ is p-injective, where $\hat{M}$ is an injective hull of M.*

Proof. As was noted on p. 176 of [13], A is a $p.p.$ ring iff all homomorphic images of any injective left A-module are p-injective. It remains therefore to prove that 2) implies 1). Let Q be an injective left A-module, and S a submodule of Q. Then, $Q=\hat{S}\oplus T$ with an injective hull $\hat{S}$ of S and a submodule T. Since $\hat{S}/S$ is p-injective and $(T\oplus S)/S$ $(\simeq T)$ is injective, $Q/S=\hat{S}/S\oplus(T\oplus S)/S$ is p-injective. Hence, A is a left $p.p.$ ring.

Concerning rings whose singular left modules are flat, we have the next (cf. [8, Theorem 1] and [14, Theorem 9]):

Theorem 5. *The following conditions are equivalent*:

1) *A is a regular ring.*

2) *Every cyclic singular left A-module is flat.*

3) *A is a left CPF-ring such that every principal left ideal is the left annihilator of an element of A.*

4) *Every cyclic singular left A-module is either p-injective or flat and every principal left ideal is the left annihilator of an element of A.*

Proof. Obviously, 1)$\Longrightarrow$2) and 1)$\Longrightarrow$3)$\Longrightarrow$4).

2)$\Longrightarrow$1) Given $b\in A$, there exists a left ideal K such that $L=Ab\oplus K$ is essential in ${}_AA$. Now, the cyclic singular left A-module A/L is flat, and so $b=bc$ with some $c\in L$ [2, Proposition 2.1]. Setting $c=ab+k\,(a\in A,\ k\in K)$, we have $b-bab=bk\in Ab\cap K=0$, namely, $b=bab$.

4)$\Longrightarrow$1) Given $b\in A$, there exists a left ideal K such that $L=Ab\oplus K$ is essential in ${}_AA$. According as A/L is flat or p-injective, the proof of 2)$\Longrightarrow$1) above or of [14, Theorem 9] applies to obtain $b=bab$ with some $a\in A$.

2. Left V-rings. Following Tominaga [10], A is called a *left p-V-ring* if every simple left A-module is p-injective. We call A a *left f-V-ring* if every simple left A-module is f-injective. Both regular rings and left

V-rings are obviously left f-V-rings but the converse is not true in either case (cf. [5], [6]). We recall here that every non-zero left ideal of a p-V-ring contains a maximal left subideal [13, Lemma 1 (ii)]. The first two theorems of this section are motivated by [9, Theorem 2. 1].

Theorem 6. *The following conditions are equivalent*:

1) *A is a left f-V-ring.*

2) *Every finitely generated left ideal of A and its maximal left subideals are intersections of maximal left ideals of A.*

Proof. 1)$\Longrightarrow$2) Let I be a finitely generated left ideal of A. Suppose there exists some $b \in I^* \backslash I$. Then $F = I + Ab$ is finitely generated, and there exists a left ideal K which is maximal with respect to $I \subseteq K \subset F$. Now, the simple module F/K is f-injective and the natural projection $F \rightarrow F/K$ can be extended to $h: {}_AA \rightarrow {}_AF/K$. Then $H = \ker h$ is a maximal left ideal of A containing I and $I^* \subseteq H$, whence it follows $F = I^* \cap F \subseteq H \cap F = K$. This contradiction shows $I^* = I$. Now, let L be a maximal left subideal of I. Suppose there exists some $c \in L^* \backslash L$. Since $L^* \subseteq I$ and L is a maximal left subideal of I, we have then $L + Ac = I (= L^*)$. Now, by making use of the above argument for the natural projection $I \rightarrow I/L$, one readily obtains a contradiction $I \subseteq L$.

2)$\Longrightarrow$1) Let M be a simple left A-module, F a finitely generated left ideal, and $g: {}_AF \rightarrow {}_AM$ a non-zero homomorphism. Since $G = \ker g$ is a maximal left subideal of F, by hypothesis there exists a maximal left ideal L of A such that $G \subseteq L$ but $F \not\subseteq L$. Then there holds $A/G = F/G \oplus L/G$, and it is easy to see that g can be extended to A.

The next extends [13, Remark 3].

Corollary 7. *If A is regular then every finitely generated left ideal and its maximal left subideals are intersections of maximal left ideals of A.*

We now give two characteristic properties of left V-rings, the second extending [14, Corollary 4] (cf. also [11, Corollary 6]).

Theorem 8. *The following conditions are equivalent*:

1) *A is a left V-ring.*

2) *A is a left p-V-ring and every left ideal I of A is a two-sided ideal of I^*.*

3) *Every minimal left ideal of A is injective and every cyclic singular left A-module is semi-simple.*

Proof. By [9, Theorem 2. 1], 1) implies 2) and 3).

2) $\Longrightarrow$ 1) Let I be a left ideal of A. Suppose there exists some $b \in I^* \backslash I$, and set $T = I + Ab$. There exists a left ideal K which is maximal with respect $I \subseteq K \subset T$. Then the simple left A-module T/K is p-injective, and the natural projection g: $Ab \rightarrow T/K$ can be extended to $h: {}_AA \rightarrow {}_AT/K$; $x \mapsto xd + K$ with some $d \in T$. Since $Id \subseteq II^* \subseteq I \subseteq K$, I is contained in the maximal left ideal ker h. Hence, $I^* \subseteq \ker h$, which implies a contradiction $Ab = I^* \cap Ab \subseteq \ker h \cap Ab = \ker g$. This means $I^* = I$, proving 1).

3) $\Longrightarrow$ 1) Let ${}_AM$ be simple, L an essential left ideal of A, and $h: {}_AL \rightarrow {}_AM$ a non-zero homomorphism. Obviously, $H = \ker h$ is a maximal left subideal of L. If H is not essential in ${}_AL$ then M is isomorphic to some minimal left ideal which is injective by hypothesis. On the other hand, if H is essential in ${}_AL$, then so is it in ${}_AA$ and there holds $H^* = H$ and $L^* = L$. Since $H \subset L$, there exists a maximal left ideal J of A such that $H \subseteq J$ but $L \not\subseteq J$. Now, it is easy to see that $J \cap L = H$ and $J + L = A$. Hence, h can be extended to $g: {}_AA \rightarrow {}_AM$.

In [8, p. 114, Query (b)], it is asked which left V-rings are regular. The next proposition shows that left V-rings whose essential right ideals are two-sided are regular.

Proposition 9. *If every essential right ideal of A is two-sided, then the following conditions are equivalent:*

1) *A is a regular ring.*
2) *A is a left CPP-ring.*
3) *A is a left p-V-ring.*
4) *A is fully left idempotent.*
5) *Every cyclic semi-simple, singular right A-module is p-injective and flat.*

Proof. Obviously, 1) implies 2) through 5). 2) implies 4) by Theorem 2, and 3) does 4) by [13, Lemma 1].

4) $\Longrightarrow$ 1) For any $b \in A$, $R = bA \oplus K$ is essential in A_A with some right ideal K. Since R is an ideal and $b \in (AbA)b \subseteq Rb$, we have $b = (ba + k)b$ with some $a \in A$ and $k \in K$. Then, $b - bab = kb \in K \cap bA = 0$, proving 1).

5) $\Longrightarrow$ 1) Let R be an arbitrary essential right ideal of A. Patterning after the proof 2) $\Longrightarrow$ 1) of Theorem 8, one will easily see that A/R is semi-simple and therefore flat. Then, A is regular by Theorem 5.

Question. Is a left V-ring whose essential left ideals are two-sided regular ?

The proof 4) $\Longrightarrow$ 1) of Proposition 9 together with Lemma 1 (2) and (3) yields the following :

Corollary 10. *If A is a left CPF-ring whose essential right ideals are two-sided, then every principal projective left ideal is a direct summand of $_AA$.*

As a combination of [9, Theorem 2. 1], Theorem 8 and Proposition 9, we readily obtain

Corollary 11. *If every essential right ideal of A is two-sided, then the following conditions are equivalent:*

1) *A is a regular, left V-ring.*

2) *Every minimal left ideal of A is injective and every cyclic singular left A-module is semi-simple.*

Corollary 12. *If every essential one-sided ideal of A is two-sided, then the following are equivalent:*

1) *A is a regular ring whose minimal right ideals are injective.*

2) *A is a right V-ring.*

Finally, we return to strongly regular rings.

Theorem 13. *The following conditions are equivalent:*

1) *A is a strongly regular ring.*

2) *A is a reduced left p-V-ring whose essential left ideals are two-sided.*

3) *A is a right duo, left V-ring.*

4) *A is a right duo, left p-V-ring.*

5) *A is a reduced left CPF-ring whose essential left ideals are two-sided.*

6) *Every simple left A-module is either p-injective or flat and A is a reduced ring whose essential left ideals are two-sided.*

Proof. Obviously, 1) $\Longrightarrow$ 2) $\Longrightarrow$ 6), 1) $\Longrightarrow$ 5) $\Longrightarrow$ 6), and 1) $\Longrightarrow$ 3) $\Longrightarrow$ 4). By Proposition 9, 4) implies 1). Finally, we shall show that 6) implies 1). To our end, it suffices to prove that $A = Aa + l(a)$ for every $a \in A$. Suppose $I = Aa + l(a) \neq A$ for some a. Let J be a maximal left ideal containing I. First we claim that J is an essential (two-sided) ideal. In fact, if not, $J = Ae$ with an idempotent $e \neq 1$, whence it follows a contradiction $1 - e \in r(J) \subseteq r(a) = l(a) \subseteq J$. If A/J is p-injective, considering $f: Aa \to A/J$ defined by $xa \mapsto x + J$, we can find an element $c \in A$ such that $1 + J = ac + J$. Since J is an ideal, this implies a contradiction $1 \in J$. While, if A/J is flat then by [2, Proposition 2. 1] there exists an element d of J with $a = ad$, and $1 - d \in r(a) = l(a) \subseteq J$, which is a contradiction.

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