COVERS OF ABELIAN GROUPS

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Introduction. Recently papers have appeared in the literature [4, 5, 9] dealing with rings and finite unions of subrings or ideals. This paper deals with abelian groups and fiinite unions of subgroups.

If G is an abelian group and $G = H_1 \cup \cdots \cup H_n$ where each H_i is a proper subgroup of G and n is finite, the set $\{H_i\}$ is called a *cover* of G. In this paper we assume all groups are abelian and that a cover ceases to be a cover if any member of it is omitted. We call a cover *maximal* if no member of it has a cover. If groups G and G' have covers C and C', respectively, then C and C' are said to be *isomorphic* if the members of C are isomorphic (as groups) to the members of C' in a one-to-one manner. An element or cyclic subgroup of a group G is called *special* if it is in exactly one cyclic subgroup of G. We ask the questions: (1) What groups have maximal covers? (2) If two groups have isomorphic maximal covers, are the groups isomorphic? (3) If a group has no cover, what is its structure? The answers are found in the theorems.

1. General facts. Let a group G have the cover $\{H_i\}_1^n$. we have the following facts.

(a) The integer n cannot be 1 or 2.
(b) If $n=3$, $G/\cap H_i$ is the Klein 4-group.
(c) A necessary and sufficient condition for a group G to have a cover is that G/pG is not cyclic for some prime p.
(d) G/H_i is finite.
(e) If the cover is maximal and H is a subgroup of finite index in some H_j, then H_j/H is cyclic.
(f) If G is finite and the cover is maximal, then $\cap H_i$ is the direct sum of the primary components of G which are cyclic.
(g) If the cover is maximal, it is unique.

The proof of (a) is trivial. The proofs of (b), (c), and (d) may be found in [3], [1], and [7] (or in [8], page 227), respectively. To show (e), suppose H_j/H is not cyclic. Since it is finite, it has a non-cyclic primary component and, by (c), a cover, say $\{K_k/H\}$, $K_k \subset G$. Then $\{K_k\}$ covers H_j which contradicts the maximality of $\{H_i\}$. In (f), since G is finite, its maximal cover consists of its special subgroups. A primary component of

G which is cyclic is clearly contained in each special subgroup of G. Conversely, if x is an element of G not contained in the direct sum of the primary components of G which are cyclic, then G has a special subgroup not containing x. To prove (g), suppose $\{H_i\}$ and $\{K_i\}$ are two maximal covers of G. Let $H = \cap H_i$ and $K = \cap K_i$. Since H and K have finite index in G, so does $H \cap K$. Therefore $G/H \cap K$ is finite and has covers $C = \{H_i/H \cap K\}$ and $C' = \{K_i/H \cap K\}$. By the maximality of $\{H_i\}$ and $\{K_i\}$, C and C' are maximal covers of G/H. Each consists of the special subgroups of G/H and $C = C'$. For some arrangement of indices, then, $H_i = K_i$ for each i.

2. Finite groups. Finite non-cyclic groups have maximal covers (consisting of their special subgroups) and for these groups we seek an answer to question 2 of the introduction. The number of special elements of order m in a finite group G is designated by $S_G(m)$. We will provide formulae to determine this number for an arbitrary finite group G and postive integer m. We will then use these formulae to prove a theorem.

Formula 1. *Let G be a p-primary finite group and, for $n > 0$, let r_n equal the number of cyclic subgroups of order p^n in some decomposition of G into a direct sum of cyclic subgroups. Set $r_0 = 0$. Then, for $k > 0$*

$$S_G(p^k) = \Big(\prod_{n<k} p^{(n-1)r_n} \prod_{n>k} p^{(k-1)r_n}\Big)\Big(\prod_n p^{r_n} - \prod_{n<k} p^{r_n} - \prod_{n>k} p^{r_n} + 1\Big).$$

Proof. Suppose $p^M G = 0$ and $G = B_0 \oplus \cdots \oplus B_M$ is the decomposition of G where each B_n is a direct sum of r_n cyclic subgroups of order p^n. Suppose g is a special element in G of order p^k, $k > 0$, and let $g = x + y + z$ where x, y, z are members of $\bigoplus_{n<k} B_n$, B_k, $\bigoplus_{n>k} B_n$, respectively. We observe that y or x must be special in G. Case 1. Assume that y is special in G. Then $o(y) = p^k$ and the number of choices for y is $|B_k| - |pB_k|$. There are $|\bigoplus_{n<k} B_n|$ choices for x and, since $o(z) \leq p^k$, there are exactly $|\bigoplus_{n>k} p^{n-k} B_n|$ choices for z. The number of choices for g is the product of these three numbers which is, by computation,

$$\prod_{n<k} p^{nr_n}(p^{kr_k} - p^{kr_k - r_k}) \prod_{n>k} p^{kr_n}.$$

Case 2. Assume y is not special in G and hence x is. Then y is in pB_k and $o(z) = p^k$. The number of choices for x, y, z is, in turn, $\prod_{n<k} |B_n| - \prod_{n<k} |pB_n|$, $|pB_k|$, and $\prod_{n>k} |p^{n-k}B_n| - \prod_{n>k} |p^{n-k+1}B_n|$. The number of choices for g is the product of these numbers which equals

$$(\prod_{n<k} p^{nr_n} - \prod_{n<k} p^{nr_n - r_n})\, p^{kr_k - r_k} (\prod_{n>k} p^{kr_n} - \prod_{n>k} p^{kr_n - r_n}).$$

Adding the results of the two cases and simplifying, we obtain the formula.

Formula 2. *If a finite group G is the direct sum of t P_i-components G_i for distinct primes $p_1, \cdots, p_t$ and $m = p_1^{k_1} \cdots p_t^{k_t}$, $k_i \geq 0$, then $S_G(m)$ equals the product $S_{G_1}(p_1^{k_1}) \cdots S_{G_t}(p_t^{k_t})$.*

Proof. This follows from the observation that an element g is special in G exactly if $g = x_1 + \cdots + x_t$ where each x_i is special in G_i.

Theorem 1. *Two finite primary groups are isomorphic if and only if they are cyclic of the same order or they have isomorphic maximal covers. The restriction to primary groups is necessary.*

Proof. Isomorphic finite groups are cyclic of like order or have isomorphic maximal covers by Fact (g). Suppose, then, G and G' are two finite p-primary groups with isomorphic maximal covers C and C', respectively. Let M be the least positive integer such that $p^M G = 0 = p^M G'$. Choose a set of ranks $\{r_n\}$ for G as in Formula 1, and, similarly, a set of ranks $\{s_n\}$ for G'. We will show $r_n = s_n$ for each n. Since C and C' consist of the special subgroups in G and G', these groups have the same number of special subgroups of order p^k for each k. Since a special subgroup of order p^k contains exactly $p^k - p^{k-1}$ special elements, $S_G(p^k) = S_{G'}(p^k)$ for each k. We now use these equations to show $r_n = s_n$ for all n. We use induction beginning with $n = M$. By our choice of M, $S_G(p^M) = S_{G'}(p^M) \neq 0$. If we make the substitutions in this equation indicated by Formula 1, we obtain an equation of the form $p^a(p^{r_M} - 1) = p^b(p^{s_M} - 1)$ for some integers a and b. Thus, $r_M = s_M$. Assume $r_n = s_n$ for $n > N$ and that $S_G(p^N) = S_{G'}(p^N) \neq 0$ (otherwise $r_N = s_N = 0$). By Formula 1, we have

$$p^a(\prod_n p^{r_n} - \prod_{n<N} p^{r_n} - \prod_{n>N} p^{r_n} + 1) = p^b(\prod_n p^{s_n} - \prod_{n<N} p^{s_n} - \prod_{n>N} p^{s_n} + 1)$$

for some integers a and b. Now, if $S_G(p^n) = S_{G'}(p^n) = 0$ for all $n < N$, then $\prod_{n<N} p^{r_n} = 1 = \prod_{n<N} p^{s_n}$, $p^a \prod_{n>N} p^{r_n}(p^{r_N} - 1) = p^b \prod_{n>N} p^{s_n}(p^{s_N} - 1)$, and $r_N = s_N$. However, if $S_G(p^n) = S_{G'}(p^n) \neq 0$ for some $n < N$, then $\prod_{n<N} p^{r_n} \neq 1 \neq \prod_{n<N} p^{s_n}$ and $\prod_n p^{r_n} - \prod_{n<N} p^{r_n} - \prod_{n>N} p^{r_n} + 1 = \prod_n p^{s_n} - \prod_{n<N} p^{s_n} - \prod_{n>N} p^{s_n} + 1$. We cancel the two right-hand terms on each side and obtain $\prod_{n<N} p^{r_n}(\prod_{n\geq N} p^{r_n} - 1) = \prod_{n<N} p^{s_n}(\prod_{n\geq N} p^{s_n} - 1)$. Thus $\sum_N^M r_n = \sum_N^M s_n$ and $r_N = s_N$. By induction, $r_n = s_n$ for all n and $G \cong G'$.

We verify the second sentence of the theorem by an example. For positive integers n and m, let Z_n^m be the direct sum of m copies of the integers modulo n. Let $G=Z_2^5\oplus Z_5^1$ and $G'=Z_2^1\oplus Z_5^3$. By Formulae 1 and 2, $S_G(10)=(2^5-1)(5-1)=124=(2-1)(5^3-1)=S_{G'}(10)$, yet G is not isomorphic to G'.

Remarks. 1) In Formula 1, S_G is evaluated in terms of a particular direct sum decomposition of G, but S_G is, by definition, independent of this decomposition. 2) The proof of Theorem 1 yields a new (albeit cumbersome) proof of the invariance of ranks of a finite primary group under different decompositions into direct sums of cyclic subgroups. Suppose G and G' are finite p-primary groups and that we have obtained particular sets of rank $\{r_n\}$ and $\{s_n\}$ for G and G', respectively. If $G=G'$, then, for each n, $S_G(p^n)=S_{G'}(p^n)$ and, by our proof, $r_n=s_n$. 3) The example we gave relies on the fact that $1+2+2^2+2^3+2^4=31=1+5+5^2$. There is no other known example of a prime number which can be expressed as a finite power series in two different ways.

3. Abelian groups in general. We now respond to the first two questions of the introduction for groups in general.

Theorem 2. *An abelian group G has a maximal cover iff it has a decomposition $G=H\oplus A$ where the order of A is positive but finite and, for every prime p, either $pA=A$ and H/pH is cyclic or $pH=H$ and A/pA is not cyclic.*

Proof. 1) Suppose G has a maximal cover $\{H_i\}$ and $H=\cap H_i$. We wish to show that H is a direct summand of G. Since G/H is a finite direct sum of cyclic groups, it will suffice to show that H is pure in G ($p^kG\cap H=p^kH$ for every prime p and positive integer k). Set $\overline{G}=G/H$. Suppose p is a prime such that $p\overline{G}=\overline{G}$. Suppose $p^kg\in H$, for some $g\in G$ and positive k. Since $(|\overline{G}|, p)=1$, there is an integer r such that $rg\in H$ and $(r,p)=1$. Let a, b be integers so that $ar+bp^k=1$. Then $g=(ar+bp^k)g=arg+bp^kg\in H$ and $p^kG\cap H=p^kH$. Suppose, then, p is a prime such that $p\overline{G}\neq\overline{G}$. We observe that $\overline{G}=G/H$ has a maximal cover $\{H_i/H\}$, so that $\cap(H_i/H)=0$, and, by Fact (f), that the p-component of G/H is non-cyclic. Suppose $H\neq pH$. Then H has a subgroup K, where $|H/K|=p^k$, $k>0$, and G/K has a maximal cover $\{H_i/K\}$ with $\cap(H_i/K)=H/K$. But, since $|H/K|$ is a power of p, $\cap(H_i/K)=0$ by Fact (f). Therefore $pH=H$ and, as a result,

$p^kG\cap H=p^kH$ for all k. Since H is pure in G, for some subgroup A, $G=H\oplus A$. If p is a prime such that $p\overline{G}\neq\overline{G}$, then A/pA is not cyclic and $pH=H$. If $p\overline{G}=\overline{G}$, then $pA=A$ and we now show H/pH is cyclic. If not, H has a subgroup K such that $|H/K|=p^k$ for some positive k, and H/K is non-cyclic. Now G/K has a maximal cover $\{H_i/K\}$ with $\cap(H_i/K)=H/K$. But H/K is non-cyclic and, by Fact (f), not in $\cap(H_i/K)$. Therefore, H/pH is cyclic. The conditions of the theorem are satisfied. 2) Suppose $G=H\oplus A$ with the properties stated in the theorem. For some subgroups $H_i\subset G$, G/H has a maximal cover $\{H_i/H\}$. Each $H_i=H\oplus C$ for some cyclic subgroup C of A. For each prime p, $H_i/pH_i\cong H/pH+C/pC$ which is cyclic. Therefore, no H_i has a cover and the cover $\{H_i\}$ of G is maximal.

From Theorems 1 and 2, we obtain the following:

Theorem 3. *If groups G and G' have maximal covers and G/pG and G'/pG' are cyclic for all primes p except one, then $G\cong G'$ iff their maximal covers are isomorphic. The restriction on primes is necessary.*

Proof. It suffices to show that, if G and G' have the given restrictions on primes and isomorphic maximal covers $\{H_i\}$ and $\{H_i'\}$, then $G\cong G'$. Let $G=H\oplus A$, $G'=H'\oplus A'$ with appropriate properties, as indicated in Theorem 2. Then $H_i=H\oplus C_i$ and $H_i'=H'\oplus C_i'$ with $C_i\subset A$, $C_i'\subset A'$. Now for fixed prime q, C_i and C_i' are q-primary anc finite while $qH=H$ and $qH'=H'$. Since $H_i\cong H_i'$ for each i, it follows that $C_i\cong C_i'$ and $H\cong H'$. Therefore G/H and G'/H' have isomorphic maximal covers $\{H_i/H\}$ and $\{H_i'/H'\}$. By Theorem 1, then, $G/H\cong G'/H'$, $A\cong A'$, and $G\cong G'$.

4. Groups without covers. A group G is without cover iff G/pG is cyclic for each prime p. We now characterize these groups further. If G is such a group, $G=D\oplus R$ where D is divisible ($pD=D$ for each p), R is reduced (it has no non-zero divisible subgroups), and R/pR is cyclic for each prime p. Let us assume, then, that G is reduced and G/pG is cyclic for each p. Let $\Pi=\prod_p(\prod_n G/p^nG)$, the complete direct sum of the groups G/p^nG where p and n range over all primes and positive integers respectively. Let ϕ be the natural map from G to Π. We claim ϕ is injective and that $\phi(G)$ is pure in Π. We first show that the kernel ϕ, $\bigcap_p\bigcap_n p^nG$, equals 0. Suppose $x\neq 0$ is a member of this subgroup. For fixed p, since G is reduced and G/pG is cyclic, the p-primary component of the torsion subgroup of G is cyclic, and $G=A\oplus B$ where $p^NA=0$ for some $N\geq 0$ and

B has no elements of order p. Since $x \in \bigcap_n p^n G$, $x \in B$ and B contains a set of elements $\{x_n\}$ such that $x = p^n x_n$ for all positive n. Since $p^n(x_n - p x_{n+1}) = 0$ and B has no elements of order p, $x_n = p x_{n+1}$ for each n. We can find a similar set of elements for each prime p and the subgroup generated by the elements in these sets is isomorphic to the rational numbers. However, G is reduced has no such subgroup. Therefore, $\bigcap_n \bigcap_p p^n G = 0$ and ϕ is injective. The proof that $\phi(G)$ is pure in Π is straightforward (see, for example, Lemma 30. 3 or Theorem 39. 5 of [2]). We now examine more closely how $\phi(G)$ sits in Π. For each prime p, let π_p be the projection of Π onto $\prod_n G/p^n G$. Suppose first that p is a prime for which G has elements of order p. Then $G = A \oplus B$, as above, with A cyclic and p-primary, and $pB = B$. Therefore, B is the kernel of $\pi_p \phi$ and $\pi_p \phi(G) \cong A$. Secondly, let p be a prime such that $G \neq pG$ but G has no elements of order p. We claim that $\prod_n G/p^n G$ contains a copy of the p-adic integers which, in turn, contains $\pi_p \phi(G)$. For $n \leq m$, let π_n^m map $G/p^m G$ onto $G/p^n G$ by sending $g + p^m G$ to $g + p^n G$. Then $\{G/p^m G;\ \pi_n^m\}$ forms an inverse system and the inverse limit, $\varprojlim G/p^n G$, consists of all vectors $(\cdots, a_m, \cdots)$ in $\prod G/p^n G$ such that $\pi_n^m a_m = a_n$, $n \leq m$ (see [2], Vol. I, page 60, for details). Since G/pG is cyclic of order p and G has no elements of order p, each $G/p^n G$ is cyclic of order p^n. As a result, the subgroup $\varprojlim G/p^n G$ of $\prod G/p^n G$ is isomorphic to the p-adic integers (see [2], Vol. I, page 62, for a proof). Now, if $x \in G$ and $\pi_p \phi(x) = (\cdots, x_m, \cdots)$, then $\pi_n^m x_m = x_n$, $n \leq m$, since $x_i = x + p^i G$ for each i. Therefore, $\pi_p \phi(G) \subseteq \varprojlim G/p^n G$. We have proved half of the following theorem.

Theorem 4. *A group G has the property: G/pG is cyclic for each prime p iff G is a pure subgroup of a group of the form $D \oplus \prod_{p \in J} K_p$, J a set of distinct primes, where D is divisible and each K_p is either cyclic and p-primary or isomorphic to the p-adic integers.*

Proof. Necessity was established above. Suppose, then, G is a pure subgroup of some $K = D \oplus \prod K_p$. We must show G/qG is cyclic for each prime q. Now $K_q/qK_q \cong K/qK \supseteq (G, qK)/qK \cong G/G \cap qK = G/qG$ by the purity of G in K. If K_q is cyclic, then G/qG is cyclic. If K_q is isomorphic to the q-adic integers, it is well known that K_q/qK_q is cyclic (e. g., see Th. 88. 1 of [2]). Again, G/qG is cyclic, and the proof is complete.

A further discussion of torsion-free groups of finite rank with the property of Theorem 4 may be found in [6].

BIBLIOGRAPHY

[1] Advanced Problem 5924: Amer. Math. Monthly, **81** (1974), 1036.
[2] L. FUCHS: Infinite Abelian Groups, Vol. 1, 1970; Vol. 2, 1972, Academic Press, N.Y.
[3] S. HABER and A. ROSENFELD: Groups as unions of proper subgroups, Amer. Math. Monthly **66** (1959), 491—494.
[4] K. KISHIMOTO and K. MOTOSE: A covering theorem for simple rings, J. Fac. Sci. Shinshu Univ. **5** (1970), 107—108.
[5] M. ÔHORI: On finite unions of subrings, Math. J. Okayama Univ. **19** (1976), 45—50.
[6] C. E. MURLEY: The classification of certain classes of torsion free groups, Pacific J. Math. **40** (1972), 647—665.
[7] B. H. NEUMANN: Groups covered by permutable subsets, J. London Math. Soc. **29** (1954), 236—248.
[8] B. H. NEUMANN: Groups covered by finitely many cosets, Publ. Math. Debrecen **3** (1955), 227—242.
[9] P. QUARTARARO and H. S. BUTTS: Finite unions of ideals and modules, Proc. Amer. Math. Soc. **52** (1975), 91—95.

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